MAT249, 2021, Day 3, Infinite Groups

(1) **Fall 2020 Problem 1:** Let k be a field, and let $B_n \subset GL_n(k)$ be the subgroup of upper triangular matrices with ones on the diagonal. Prove that B_n can be expressed as a semidirect product

$$B_n \cong H_1 \ltimes (H_2 \ltimes \cdots (H_{l-1} \ltimes H_l) \cdots),$$

where each group H_i is isomorphic to a vector space over k with the group law being addition.

(2) **Fall 2016 Problem 1:** Let F_n be the free group on n generators with $n \geq 2$. Prove that the center $Z(F)$ of F is trivial.

(3) **Fall 2015 Problem 5:** Show that two free groups are isomorphic if and only if they have equal ranks.

(4) **Fall 2014 Problem 1:** Let G_1, G_2 be finite index subgroups of a group G. Show that the intersection $G_1 \cap G_2$ also has finite index in G.

(5) **Fall 2014 Problem 3 and two other exams:** Show that there are no nontrivial homomorphisms from $\mathbb{Q}$ to a finitely generated Abelian group.

(6) **Fall 2013 Problem 1:** Let $G \subset M_n(\mathbb{C})$ be a group of complex n by n matrices. Let V be the linear span of G, and let $V^\times$ be the set of invertible elements of V. Show that $V^\times$ is also a group.

(7) **Fall 2012 Problem 2:** Let G be an abelian group with n generators. Show that every subgroup $H \subset G$ has a generating set consisting of at most n elements.

(8) **Fall 2010 Problem 3:** Show that the (multiplicative) group of $n \times n$ upper-triangular matrices (with real entries), having diagonal elements that are non-zero, is solvable.

(9) **Winter 2009 Problem 5:** Find all homomorphisms from $(\mathbb{Q}, +)$ to $(\mathbb{Q}_{>0}, \times)$.

(10) **Winter 2008 Problem 1:** Show that every finitely generated group contains only finitely many subgroups of index n.

(11) **Winter 2006 Problem 6:** Show that if H is a subgroup contained in every nontrivial subgroup of G then H is contained in the center $Z(G)$.

(12) **Winter 2005 Problem 3:** Show that $\mathbb{Q}$ contains no proper subgroups of finite index.

(13) **Fall 2004 Problem 2:** Show that no infinite simple group has proper subgroups of finite index.

(14) **Winter 2004 Problem 11:** Show that $\mathbb{Q}$ is not a nontrivial direct sum.