MAT 249, 2021, Day 4, Fields

(1) **Fall 2020 Problem 5:** Let F be a field of characteristic p that is not prefect (so, not every element in F is a p-th power of another). Show that there exist inseparable irreducible polynomials in $F[x]$.

(2) **Fall 2020 Problem 6:** Suppose that E is a Galois extension of a field F such that $[E, F] = 5^3 \cdot 43^2$. Prove that there exist fields K_1 and K_2 lying strictly between F and E with the following properties:
   (a) each K_i is a Galois extension of F
   (b) $K_1 \cap K_2 = F$;
   (c) $K_1 K_2 = E$.

(3) **Fall 2019 Problem 3:** Let $\zeta = \exp(2\pi i/5)$ a primitive fifthroot of unity.
   (a) Find the degree of the field extension $[\mathbb{Q}[\zeta] : \mathbb{Q}]$, and describe a basis of $\mathbb{Q}[\zeta]$ as a vectorspace over $\mathbb{Q}$.
   (b) Show that $G = \mathrm{Aut}(\mathbb{Q}\zeta]/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_4$.
   (c) Describe the action of G as matrices in the basis from the first part.
   (d) Let $H \subset G$ be the order two subgroup $\{1, a^2\}$, where a is a generator of G. Describe the field F of elements of $\mathbb{Q}(\zeta)$ preserved by his subgroup. What's the degee $[F : \mathbb{Q}]$?

(4) **Fall 2018 Problem 3:** Let $f = x^4 - 14x^2 + 9 \in \mathbb{Q}[x]$. Compute the Galois group of f.

(5) **Fall 2017 Problem 1:** Suppose that $F \subseteq K$ is an inclusion of fields and let $\alpha, \beta \in K$ be two elements which are algebraic over F. Show that $\alpha + \beta$ is also algebraic over F.

(6) **Fall 2017 Problem 2:** Let $f \in \mathbb{Q}[x]$ be the minimal polynomial of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$, and let K be the splitting field for f over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$ and what is $\mathrm{Gal}(K/\mathbb{Q})$? (Note that you are not required to find f.)

(7) **Fall 2016 Problem 5:** Let F be a finite field and let L be a subfield of F generated by elements of the form x^3 for all $x \in F$. Prove that if $L \neq F$, the F has exactly 4 elements.

(8) **Fall 2015 Problem 6:** Find the $\mathbb{Q}$ dimension of the splitting field over $\mathbb{Q}$ of $x^5 - 3$.

(9) **Fall 2013 Problem 5:** Is it possible to have a field extension $F \subseteq K$ with $[K : F] = 2$, where both fields F and K are isomorphic to the field $\mathbb{Q}(x)$?

(10) **Fall 2013 Problem 6:** Compute $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.

(11) **Fall 2009 Problem 2:** The field extensions $\mathbb{Q}(\sqrt{\sqrt{2}})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are both Galois. Show that $\mathbb{Q}(\sqrt{\sqrt{2}})/\mathbb{Q}$ is not Galois.

(12) **Fall 2004 Problem 6:** Construct infinitely many nonisomorphic quadratic extensions of $\mathbb{Q}$ and use these to show that the Abelianization of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is not finitely generated.