MAT 249, 2021, Day 5, Ideals

(1) **Fall 2020 Problem 2.** Let D be a Principal Ideal Domain (PID). Show that every proper ideal in D is a product $\mathbf{m}_1\mathbf{m}_2\cdots\mathbf{m}_k$ of maximal ideals, which are uniquely determined up to order.

(2) **Fall 2018 Problem 5.** Solve the following questions:
   (a) If F is a field, prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over F.
   (b) Show that $f(x) = x^2+2x+2$ is irreducible in $\mathbb{Q}[x]$, and find the inverse of $1+x$ in $\mathbb{Q}[x]/(f(x))$.

(3) **Fall 2017 Problem 4.** Find all maximal ideals in the ring $\mathbb{F}_7[x]/(x^2+1)$ and the ring $\mathbb{F}_7[x]/(x^3+1)$.

(4) **Fall 2016 Problem 4.** Let $I, J \subseteq R$ be ideals in a principle ideal domain R. Prove that $I + J = R$ iff $IJ = I \cap J$.

(5) **Fall 2015 Problem 4.** Let I denote the ideal in the ring $\mathbb{Z}[x]$ generated by 5 and x^3+x+1. Is I a prime ideal?

(6) **Fall 2014 Problem 4.** List all ideals of $\mathbb{F}_p[x]/(x^2+x-6)$ when
   (a) $p = 7$
   (b) $p = 5$.

(7) **Fall 2012 Problem 4.** Let R be a commutative ring and I an ideal in R. Prove or disprove: The set $\sqrt{I} = \{a \in R : \exists n \in \mathbb{N}, n > 0, a^n \in I\}$ is an ideal.

(8) **Fall 2010 Problem 4.** Consider the ring $R = \mathbb{Z}[x]$. Give an example, with a proof, of an ideal in R which is not principal and of an ideal that is not prime.