MAT 249, 2021, Day 6, Group Representations

(1) **Fall 2019 Problem 2.** Show that the quaternion group $Q_8 = \langle i, j, k : i^2 = j^2 = k^2 = ijk = i^6 \rangle$ of order 8 has a two-dimensional irreducible representation $Q_8 \to GL_2(\mathbb{C})$. (This corrects the original presentation).

(2) **Fall 2018 Problem 6.** Let V be the subspace of $\mathbb{C}^3$ spanned by $v_1, v_2 = (1, -1, 0), (0, 1, -1)$ which is an invariant subspace under the permutation action of S_3, and so gives a two dimensional representation $\rho : S_3 \to \mathrm{GL}(V)$.
   (a) Write down the matrices of $\rho(\sigma)$ in this basis.
   (b) Describe a Hermitian inner product $\langle , \rangle$ on V in the basis v_1, v_2 which is G-invariant. i.e. $\langle \rho(g)u, \rho(g)v \rangle = \langle v, u \rangle$.
   (c) Describe the tensor product $V \otimes_R V$, as an R-module where $R = \mathbb{C}[H]$ and $H \subset S_3$ is the subgroup $\{1, (123), (132)\}$.

(3) **Fall 2014 Problem 5.** Let ρ be a representation of a finite group G on a vector space V and let $v \in V$.
   (a) Show that averaging $\rho_g(v)$ over G gives a vector $\overline{v} \in V$ which is fixed by G.
   (b) What can you say about this vector when ρ is an irreducible representation?

(4) **Fall 2013 Problem 3:** The dihedral group D_4 acts as the symmetries of a square in the plane $\mathbb{R}^2$ with coodinates x and y. Suppose tht the corners of this square are at $(\pm 1, \pm 1)$. Then D_4 acts linearly, and it therefore has an induced action on the vector space V_n of homogeneous polynomials in x and y of degree n. Find the character of V_n viewed as a representation of D_4. (Note: the character will depend on n.)

(5) **Fall 2012 Problem 6.** Consider the dihedral group $D_4 = \langle r, s : s^2 = r^4 = 1, rs = sr^{-1} \rangle$ of order 8.
   (a) Find the conjugacy classes of D_4.
   (b) Find the character table of D_4.

(6) **Fall 2011 Problem 4:** Let G be the subgroup of S_{12} generated by $a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$ and by $b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)$. Find the order of G, the number of conjugacy classes of G, and the character table of G.

(7) **Fall 2009 Problem 6:** Find an example of a nontrivial finite dimensional representation of a group over some field for which the character is identically zero.

(8) **Fall 2007 Problem 1:** Show that every matrix in the image of a complex representation of a finite group is diagonalizable.

(9) **Winter 2007 Problem 3:** Consider the following 5 by 5 character table for the finite group G in which a is a primitive 5th root of unity:

1	1	1	1	1
3	-1	0	$a^3 + a^2 + 1$	$a^4 + a + 1$
3	-1	0	$a^4 + a + 1$	$a^3 + a^2 + 1$
4	0	1	-1	-1
5	1	-1	0	0

(a) Show that G is simple and of order 60 and find the orders of its conjugacy classes.
(b) Find the decomposition into irreducible representations of the tensor produt of the two 3-dimensional irreducible representations.

(10) **Fall 2006 Problem 5:** Define $\phi_A(X) = AXA^{-1}$ if $X \in \mathbb{C}^{2\times 2}$ and $A \in GL_2(\mathbb{C})$.
(a) Check that this defines a linear representation of $GL_2(\mathbb{C})$
(b) Show that this representation is reducible.
(c) Write down one element of each orbit of the above action and in each case find the stabilizer of that element.