Math 201A 2018 Take-Home Midterm Solutions

(1) Consider a sequence \( \{(X_i, d_i)\}_{i=0} \) of compact, diameter-1 metric spaces (that is, \( \sup_{x,y \in X_i} d_i(x, y) = 1 \) for every \( i \)) and define \( X = \prod_i X_i \) the product as a set and \( d(\{x_i\}, \{y_i\}) = \sup_i d_i(x_i, y_i) \).

Show that \( (X, d) \) is

(a) a metric space,

ANS: Check the triangle inequality: \( \forall \{x_i\}, \{y_i\}, \{z_i\} \in X, d(\{x_i\}, \{z_i\}) = \sup_i d_i(x_i, z_i) \leq \sup_i (d_i(x_i, y_i) + d_i(y_i, z_i)) \leq \sup_i d_i(x_i, y_i) + \sup_j d_j(y_j, z_j) = d(\{x_i\}, \{y_i\}) + d(\{y_j\}, \{z_j\}) \). The other properties of a metric are easier.

(b) complete,

ANS: It must be shown that every Cauchy sequence \( S = \{\{x^n_i\}\}_n \) in \( X \) converges. Here \( x^n_i \in X_i \) since \( S \) is Cauchy \( \forall \epsilon > 0 \exists N \forall n, m > N, i \) there is \( d_i(x^n_i, x^m_i) < \epsilon \). Since each \( X_i \) is a compact metric space it is also a complete metric space and since from above \( \forall i \) there is \( S_i = \{x^n_i\}_n \) Cauchy in \( X_i \) one has \( S_i \) converges in \( X_i \) to something. Call the limit \( y_i \in X_i \). Finally one must show that \( S \) converges to \( \{y_i\} \) in \( X \) which requires that the convergence of all the \( S_i \) to \( y_i \) in \( X_i \) by uniform in \( i \). This requires going back to the Cauchy property and computing: For \( \epsilon > 0 \) use the above \( N \) and for each \( i \) choose \( M_i > N \) so that \( \forall m \geq M_i \) there is \( d(x^n_i, y_i) < \epsilon \). Thus \( \forall n > N \) there is \( d(\{x^n_i\}, \{y_i\}) = \sup_i d_i(x^n_i, y_i) \leq \sup_i d_i((x^n_i, x^M_i) + d_i(x^M_i, x^n_i)) \leq 2\epsilon \).

(c) not separable,

ANS: It must be shown that no countable subset of \( X \) is dense. For each \( i \) choose \( y_i, z_i \in X_i \) with \( d(y_i, z_i) > 1/2 \) which is possible because of the diameter of \( X_i \). Set \( S = \{\{x_i\}_i | x_i \in \{y_i, z_i\}\} \subseteq X \). \( S \) is uncountable and if \( s, t \in S \) and \( s \neq t \) then \( d(s, t) > 1/2 \). Thus no countable subset of \( X \) has some point within \( 1/4 \) of each element of \( S \).

(d) not homeomorphic to the product topological space.

ANS: It suffices to show that one topological space is compact and the other is not. By the Tychonoff Theorem the product topology is compact. Choose an open cover of the other topology with no finite (or even countable) subcover as follows. If \( s \in S \subseteq X \) from the previous part set \( U_s \) to be the open ball of radius \( 1/2 \) about \( s \) and add one more set \( V \) the complement in \( X \) of all closed balls of radius \( 1/4 \) about elements \( s \in S \). The desired open cover is \( \{V\} \cup \{U_s | s \in S\} \).

(2) Find functions \( K \in C([-1, 1]^2) \) and \( f \in C(-1, 1) \) so that \( (T_K f)(x) = \int_{-1}^1 K(x, y) f(y) dy \) is finite for every \( x \in (-1, 1) \) but \( T_K f \) is not continuous at 0.

ANS: The point here is that if \( K \) restricts and then extends to \( K' \in C([-\epsilon, \epsilon] \times [-1, 1]) \) then \( T_K : C([-1, 1]) \to C[\epsilon, \epsilon] \) so either \( K \) has no such extension or \( f \) has no extension to \( C[-1, 1] \). The simplest example has \( f(y) = 1 \) the constant function and \( K(x, y) = 2x^{-2} (x + y - 1) \) if \( x + y > 1 \) and 0 otherwise. This \( K \) is a piecewise linear hat function around
y = 1 for each x becoming infinite near the point (0, 1). Computing gives $T_K f = \chi_{[0,1]}$.

(3) Show that there is a unique Borel measure $\mu$ on $[0, 1]$ with the usual metric topology so that for every non-negative integer $n$ one has $\int d\mu(x)x^n = 3 + \frac{1}{n+2}$ and compute $\mu([0, \frac{1}{2}])$.

(You may assume that positive linear functions from $C[0,1]$ to $\mathbb{R}$ are $\|\cdot\|_{[0,1]}$-continuous).

ANS: Since the polynomials are dense in $C[0,1]$ by the Weierstrass theorem and since positive linear functions are continuous (as given in the problem) there is at most one positive linear function $J : C[0,1] \to \mathbb{R}$ with $J(x^n) = 3 + \frac{1}{n+2}$ and hence by the Riesz-Markov theorem there is at most one Borel measure $\mu$ satisfying the hypotheses. To see that there is one consider the Borel measure with $\mu(A) = \int_0^1 x\chi_A(x)dx$ if $1 \notin A$ and $\mu(A) = 3 + \int_0^1 x\chi_A(x)dx$ otherwise where dx the usual Lebesgue measure. Note that $d\mu$ satisfies the hypotheses and $\mu([0, \frac{1}{2}]) = \frac{3}{8}$.

(4) For any $p, q \in (1, \infty)$ consider the vector space $X^{p,q} = L^p{\mathbb{R}} \cap L^q{\mathbb{R}}$ and the product Banach space $Y^{p,q} = L^p{\mathbb{R}} \times L^q{\mathbb{R}}$ with norm $\|\cdot\|_y$ along with the diagonal map $\Delta : X^{p,q} \to Y^{p,q}$ with $\Delta(f) = (f, f)$. $X^{p,q}$ has three norms: $\|\cdot\|_p$, $\|\cdot\|_q$ and $\|\cdot\|_{y*}$ where $\|f\|_{y*} = \|\Delta(f)\|_y$.

For which (if any) of these norms is $X$ complete?

ANS: If $p = q$ all three are equivalent and $X = L^p{\mathbb{R}}$ is complete.

If $p \neq q$ there is $f \in L^p{\mathbb{R}} - L^q{\mathbb{R}}$ and $X$ contains the simple functions which are $\|\cdot\|_p$-dense in $L^p{\mathbb{R}}$ so $(X, \|\cdot\|_p)$ is a nonclosed subset of a complete metric space and hence not complete. Similarly $(X, \|\cdot\|_q)$ is not complete.

Since the product of Banach spaces is Banach and $\Delta : X \to Y$ is a $\|\cdot\|_{y*}$-isometry to its image $(X, \|\cdot\|_{y*})$ is complete iff the image of $\Delta$ is closed.

Since the projection maps for a product Banach space are continuous, if $(\Delta f_n) \rightarrow (f, g)$ then $(f_n) \rightarrow (f)$ converges to $f$ and $\|\cdot\|_y$-converges to $g$ and by passing to a subsequence by Proposition 2.9 there is a.e. pointwise convergence of $(f_n)$ to both $f$ and $g$ so $(f, g) = (f, f) = \Delta f$ so $\Delta$ has closed image and $(X, \|\cdot\|_{y*})$ is complete.

(5) For three of the following show that there is a unique solution $u \in C[0,1]$ and find an upper bound for $u(1)$.

(a) $u(x) = \frac{1}{3} \int_0^1 e^xz u(z)dz + 4 \sin(2\pi x)$
(b) $u(x) = \frac{1}{3} \int_0^1 e^xz u(z)dz + 4 \sin(2\pi x)$
(c) $u(x) = 4 \int_0^1 e^xz u(z)dz + \frac{1}{3} \sin(2\pi x)$
(d) $u(x) = 4 \int_0^1 e^xz u(z)dz + \frac{1}{3} \sin(2\pi x)$

In each such equation is $u = T_K u + g$ or $(I - T_K)u = g$ so that if $\lim_{N \to \infty} \sum_{n=0}^{N} \|T^n_K\|_{op}$ exists then so does $\lim_{N \to \infty} \sum_{n=0}^{N} \|T^n_K\| = (I - T_K)^{-1}$ and $u = (I - T_K)^{-1}g$ is the unique solution and $u(1) \leq \|u\|_{[0,1]} \leq \|(I - T_K)^{-1}g\|_{[0,1]}$.

(a) $\|g\|_{[0,1]} = 4$ and $K(x, z) = \frac{1}{4} e^{xz}$. This is a Fredholm case so $\|T^n_K\| \leq \|K\|_{op}^n (\frac{1}{4})^n$ and hence $u(1) \leq \frac{1}{\frac{1}{4} - 4} = 10^{-4}$.

(b) $\|g\|_{[0,1]} = 4$ and $K(x, z) = \frac{1}{4} e^{xz}$ if $z < x$ and 0 otherwise. This is a Volterra case so $\|T^n_K\| \leq \frac{\|K\|_{op}^n}{n!}$ and hence $u(1) \leq e^{\frac{1}{4}4}$. 

(c) Here the Neumann series for $(I - T_K)^{-1}$ does not converge.
(d) $\|g\|_{[0,1]} = 4^{-1}$ and $K(x, z) = 4e^{xz}$ if $z < x$ and 0 otherwise. This is another Volterra case and hence $u(1) \leq \frac{4e}{4}$.

(6) Consider the absolute value function $f \in L^2[-1,1]$ with $f(x) = |x|$ and the three-dimensional subspace $D = \{(a + bx + cx^2 + dx^3) \in L^2[-1,1]\}$ of low degree polynomials.

(a) Find the $\| \cdot \|_{L^2}$-closest polynomial in $D$ to $f$.
(b) Is the same polynomial also a $\| \cdot \|_{[-1,1]}$-closest polynomial in $D$ to $f$?

Note that $D = D_{ev} \oplus D_{od} = \langle 1, x^2 \rangle \oplus \langle x, x^3 \rangle$.

(a) Since even and odd functions in $L^2[-1,1]$ form orthogonal subspaces, if $h$ is even and $k$ is odd then $\|h\|^2_{L^2} \leq \|h\|^2_{L^2} + \|k\|^2_{L^2} = \|h + k\|^2_{L^2}$. Thus $P_D f = f - F = a + cx^2 \in D_{ev}$ with $F$ orthogonal to $D$ so if $p \in D$ then $(f, p) = (P_D f, p) = (p, a + (x^2, p)c$ and taking $p = 1$ and $p = x^2$ gives $1 = 2a + \frac{2c}{3}$ and $\frac{1}{2} = \frac{2a}{3} + \frac{2c}{3}$. Solving gives $P_D f = \frac{4}{15} + \frac{16}{15}x^2$. 

(b) No. Compute $d_{[-1,1]}(x^2, f) = \sup_{x \in [0,1]} (x - x^2) = \frac{1}{4} < d_{[-1,1]}(P_D f, f) \leq P_D f(0) - f(0) = \frac{a}{4}$ so $x^2$ is $\| \cdot \|_{[-1,1]}$-closer to $f$. (There is a closest one: $\frac{1}{8} + x^2$ with distance $\frac{1}{8}$).