DO NOT TURN OVER THIS PAGE
UNTIL INSTRUCTED TO DO SO!

Write your name, student ID, and signature NOW!

NO NOTES, CALCULATORS, OR BOOKS ARE ALLOWED.
NO ASSISTANCE FROM CLASSMATES IS ALLOWED.

Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. Be organized and neat, and use notation appropriately. You will be graded on the proper use of derivative and integral notation.

Please write legibly!

<table>
<thead>
<tr>
<th>#</th>
<th>Student’s Score</th>
<th>Maximum possible Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Total points</td>
<td>100</td>
</tr>
</tbody>
</table>
1. (a) (2 points) Show that the equation \( y = e^{x^3} \) is a solution of the differential equation \( y'' - 3x^2y' - 6xy = 0 \).

\[
y' = 3x^2e^{x^3}, \quad y'' = 6xe^{x^3} + 9x^4e^{x^3}
\]

\[
y'' - 3x^2y' - 6xy = (6xe^{x^3} + 9x^4e^{x^3}) - 3x^3(3x^2e^{x^3}) - 6x(e^{x^3})
\]

\[
= 6xe^{x^3} + 9x^4e^{x^3} - 9x^4e^{x^3} - 6e^{x^3} = 0 \text{ as required.}
\]

(b) (4 points) Use any method to find the general solution of the differential equation \( x^2y'' - 2y^2 = xy^2 \). You must give \( y \) as a function of \( x \).

\[
\Leftrightarrow x^2y' = xy^2 + 2y^2 \Leftrightarrow x^2y' = y^2(x + 2) \Leftrightarrow y' = y^2\left(\frac{1}{x} + \frac{2}{x^2}\right)
\]

\[
\int \frac{1}{y^2} \, dy = \int \frac{1}{x} + \frac{2}{x^2} \, dx \Leftrightarrow -\frac{1}{y} = \ln|x| - \frac{2}{x} + C
\]

\[
\Rightarrow y = -\left(\ln|x| - \frac{2}{x} + C\right)^{-1}
\]

(c) (4 points) Use any method to find the general solution of the differential equation \( xy' - y = x(\ln(x))^3 \). You must give \( y \) as a function of \( x \).

Let \( u(x) = e^\int \frac{1}{x} \, dx = e^{\ln(x)} = x \), \( y' - \frac{1}{x} y = (\ln(x))^3 \)

\[
\text{So, } y = u(x) \int u(x) Q(x) \, dx = \frac{1}{x^2} \int x^{-1} (\ln(x))^3 \, dx
\]

Let \( u = \ln(x) \). Then, \( \frac{du}{dx} = \frac{1}{x} \) \( (C \text{ is a constant})\)

\[
\text{So, } y = \frac{1}{x^2} \int x^{-1} (\ln(x))^3 \, dx = x \int \frac{du}{dx} (u)^3 \, dx = x \int u^2 du = x\left(\frac{u^3}{3} + C\right)
\]

\[
= x\left(\frac{1}{3}(\ln(x))^3 + C\right)
\]

\[
\text{So, } y = \frac{x}{3}(\ln(x))^3 + Cx
\]
2. A solution containing 1 pound of salt per gallon flows into tank at the rate of 5 gallons per minute and the well-stirred mixture flows out of the tank at the rate of 4 gallons per minute. The tank initially holds 50 gallons of water containing no salt. Let \( S \) be the number of pounds of salt in the tank at time \( t \) minutes. So, the differential equation is 
\[
\frac{dS}{dt} = 5 - \frac{4S}{50+t}.
\]

(a) (6 points) Find the particular solution to the differential equation (i.e. solve the differential equation and determine all the constants).

\[
\begin{align*}
LHS &= \frac{d}{dt} e^{\ln(50+5)} = 4 \ln(50+t) = (50+t)^4.
\end{align*}
\]

\[
LHS = \frac{4}{50+t} \int e^{\ln(50+t)} dt = e^{\ln(50+t)} = (50+t)^4.
\]

\[
\int e^{\ln(50+t)} dt = \int (50+t)^4 dt = \frac{1}{50+t} \int (50+t)^4 dt.
\]

\[
\int (50+t)^4 dt = (50+t)^5 + C.
\]

At \( t=0 \), \( S=0 \), so. \( 0 = (50+0)^5 + C \Rightarrow C = -50^5 \)

So,
\[
S = (50+t)^4 \left( (50+t)^5 - 50^5 \right)
\]

(b) (2 points) How much salt is in the tank after 30 minutes? \( t=30 \).

\[
S = (50+30)^4 \left( (50+30)^5 - 50^5 \right) = 80^4 \left( 80^5 - 50^5 \right)
\]

(c) (2 points) How much salt will be in the tank as \( t \) becomes infinitely large?

\[
\text{As} \ t \to \infty, \ S \to \infty.
\]
3. (a) (2 points) Consider the function \( f(x) = \sqrt{x^2 + y^2 - 9} \). Find the domain of \( f \) and sketch the domain.

**Domain** = \( \{(x, y) : x^2 + y^2 - 9 \geq 0\} \)

**Range** = \([0, \infty)\)

(b) (2 points) Find the radius and center of the sphere whose equation is

\[ x^2 + y^2 + z^2 - x + 3y - 2z + \frac{5}{2} = 0. \]

\[
(x^2-x) + (y^2+3y) + (z^2-2z) + \frac{5}{2} = 0.
\]

\[
\iff (x - \frac{1}{2})^2 - \frac{1}{4} + (y + \frac{3}{2})^2 - \frac{9}{4} + (z - 1)^2 - 1 + \frac{5}{2} = 0.
\]

\[
\iff (x - \frac{1}{2})^2 + (y + \frac{3}{2})^2 + (z - 1)^2 = \frac{4}{4} + \frac{9}{4} + 1 - \frac{5}{2} = 1
\]

So, center = \( \left( \frac{1}{2}, -\frac{3}{2}, 1 \right) \) and radius = 1

(c) (6 points) Consider the surface \( z = 9 - x^2 - y^2 \). Determine and sketch the \( xy \)-trace, \( xz \)-trace, and the \( yz \)-trace for this surface.
4. (a) (2 points) Find the slopes of the surface given by \( z = \frac{x^2}{y} \) at the point \((1, 1, 1)\) in the \(x\)-direction and the \(y\)-direction.

\[
2_x = \frac{2x}{y}, \quad 2_y = -\frac{x^2}{y^2}
\]

The slope in the \(x\)-direction is \(2_x(1,1) = \frac{2(1)}{(1)} = 2\)

The slope in the \(y\)-direction is \(2_y(1,1) = -\frac{(1)^2}{(1)^2} = -1\)

(b) (4 points) Find all of the critical points of \( f(x, y) = x^3 - 3x + 2y^2 - 8y \).

(Hint: there are two critical points).

\[
f_x = 3x^2 - 3 = 0 \implies 3x^2 = 3 \implies x^2 = 1 \implies x = 1 \text{ or } x = -1.
\]

\[
f_y = 4y - 8 = 0 \implies 4y = 8 \implies y = 2.
\]

So, there are 2 critical points: \((1,2)\) and \((-1,2)\).

(c) (4 points) The function \( h(x, y) = 3x^2y - 4xy + y^2 \) has three critical points: \((0,0)\), \((0, \frac{3}{2})\), and \((\frac{3}{2}, \frac{3}{2})\). Classify each of the critical points of \( h \).

\[
h_x = 6xy - 4y, \quad h_{xx} = 6y, \quad h_{xy} = 6x - 4 = h_{yx}
\]

\[
h_y = 3x^2 - 4x + 2y, \quad h_{yy} = 2
\]

So, \( \Delta(h_{xy}) = h_{xx} \cdot h_{yy} - (h_{xy})^2 = (6y)(2)-(6x-4)^2 = 12y - (6x-4)^2 \)

<table>
<thead>
<tr>
<th>C.P.</th>
<th>( h(x,y) )</th>
<th>( \Delta )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>(12(0) - (6(0) - 4)^2 = 16 &lt; 0 )</td>
<td>-</td>
<td>saddle</td>
</tr>
<tr>
<td>((0, \frac{3}{2}))</td>
<td>(12(\frac{3}{2}) - (6(0) - 4)^2 = 0 )</td>
<td>-</td>
<td>unknown</td>
</tr>
<tr>
<td>((\frac{3}{2}, \frac{3}{2}))</td>
<td>(12(\frac{3}{2}) - (6(\frac{3}{2}) - 4)^2 = 8 &gt; 0 )</td>
<td>(6(\frac{3}{2}) = 4 &gt; 0)</td>
<td>rel min</td>
</tr>
</tbody>
</table>
5. (10 points) Use the method of Lagrange multipliers to minimize the function

\[ f(x, y, z) = x^2 + y^2 + z^2 \]

subject to the constraints

\[ x - y + 2z = 3 \text{ and } 3x + y - z = 0. \]

\[ F = x^2 + y^2 + z^2 - \lambda(x - y + 2z - 3) - \mu(3x + y - z). \]

\[ F_x = 2x - 2 - 2\lambda = 0 \quad (1) \]
\[ F_y = 2y + 2 - \lambda = 0 \quad (2) \]
\[ F_z = 2z - 2\lambda + \mu = 0 \quad (3) \]
\[ F_x = -(x - y + 2z - 3) = 0 \quad (4) \]
\[ F_y = -(3x + y - z) = 0 \quad (5) \]

\[ \Rightarrow \lambda = 2x - 3\mu \]

\[ \Rightarrow 2y + \lambda - 2\lambda = 0 \Rightarrow 2y + (2x - 3\mu) - 2\mu = 0 \Rightarrow 2x + 2y - 4\mu = 0 \Rightarrow \mu = \frac{1}{2}(x+y) \]

\[ 2x - 2\lambda + \mu = 0 \Rightarrow 2x - 2(2x - 3\mu) + \mu = 0 \Rightarrow 2x - 4x + 7\mu = 0 \Rightarrow 2x - 4x + \frac{7}{2}(x+y) = 0 \Rightarrow \frac{1}{2}x + \frac{7}{2}y + 2z = 0 \]
\[ \Rightarrow -x + 7y + 4z = 0. \quad (6) \]

\[ z = 3x + y. \]
\[ x - y + 2z - 3 = 0 \Rightarrow x - y + 2(3x + y) - 3 = 0 \Rightarrow x - y + 6x + 2y = 3 \]
\[ \Rightarrow 7x + y = 3 \quad (7) \]
\[ \Rightarrow y = -7x + 3 \]

\[ -x + 7y + 4z = 0 \Rightarrow -x + 7y + 4(3x + y) = 0 \Rightarrow -x + 7y + 12x + 4y = 0 \]
\[ \Rightarrow -11x + 11y = 0 \Rightarrow -x + y = 0 \Rightarrow x = -7x + 3 \Rightarrow x = \frac{3}{8}. \]
\[ 8x = 3 \Rightarrow x = \frac{3}{8}. \]
\[ y = -7(\frac{3}{8}) + 3 = \frac{3}{8} \text{ and } z = 3x + y = 3(\frac{3}{8}) + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}. \]

The minimum occurs at \( \left( \frac{3}{8}, \frac{3}{8}, \frac{3}{2} \right) \) with value \( f\left( \frac{3}{8}, \frac{3}{8}, \frac{3}{2} \right) = \frac{3}{8} + \frac{3}{8} + \frac{3}{2} = \frac{3}{2}. \)
6. Consider the following integral:

\[ \int_0^1 \int_{x^2}^{\sqrt{x}} 8xy \, dy \, dx. \]

(a) (3 points) Sketch the region \( R \) over which we are integrating.

(b) (4 points) Evaluate the double integral.

\[
\int_0^1 \int_{x^2}^{\sqrt{x}} 8xy \, dy \, dx = \int_0^1 \left[ 4xy^2 \right]_{x^2}^{\sqrt{x}} \, dx \\
= \int_0^1 4x(\sqrt{x})^2 - 4x(x^2)^2 \, dx = \int_0^1 4x^2 - 4x^5 \, dx \\
= \left[ \frac{4}{3}x^3 - \frac{4}{6}x^6 \right]_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}
\]

(c) (3 points) Change the order of integration of the double integral.

\[
\int_0^1 \int_{x^2}^{\sqrt{x}} 8xy \, dy \, dx = \int_0^1 \int_{y^2}^{\sqrt{y}} 8xy \, dx \, dy
\]
7. (a) (2 points) Determine the \( n \)th term (starting with \( n = 1 \)) of the following sequence: \( 2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots \)

\[
A_n = (-1)^{n+1} \frac{1}{2^{n-2}}
\]

(b) Determine whether each series converges or diverges. You must justify your answer and state which test you are using.

i. (2 points) \( \sum_{n=1}^{\infty} 5n^{-\frac{1}{2}} \).

\[
= \sum_{n=1}^{8} 5 \frac{1}{n^{\frac{1}{2}}} \quad \text{p-series test}
\]

\( \text{Diverges} \) by \( p \)-series test since \( \frac{1}{2} \leq 1 \).

ii. (3 points) \( \sum_{n=3}^{\infty} \frac{2^n}{1000n^7} \).

\( \text{Diverges} \) by \( n^{\text{th}} \)-term test since \( \lim_{n \to \infty} \frac{2^n}{1000n^7} = \infty \neq 0 \).

iii. (3 points) \( \sum_{n=5}^{\infty} \frac{3n^3}{(n+1)!} \).

\[
\lim_{n \to \infty} \left| \frac{3(n+1)^3}{(n+1)!} - \frac{3n^3}{n!} \right| = \lim_{n \to \infty} \left| \frac{3}{n} \cdot \frac{(n+1)^3}{n^3} \cdot \frac{(n+1)!}{(n+2)!} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(1+\frac{1}{n})^3}{1} \cdot \frac{1}{(n+2)} \right| = 0 < 1.
\]

\( \text{Converges} \) by ratio test.
8. (5 points) Find the radius of convergence and interval of convergence of the following power series:

\[ \sum_{n=1}^{\infty} \frac{n^2}{(n-1)!} (x + 5)^n. \]

You do NOT need to say what happens at the end points of the interval.

Ratio Test:

\[ \lim_{n \to \infty} \left| \frac{(n+1)^2}{(n+1-1)!} \cdot \frac{n^2}{(n-1)!} 
\times (x + 5) \right| \]

\[ = \lim_{n \to \infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{(n-1)!}{n!} \cdot (x + 5) \right| \]

\[ = \lim_{n \to \infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{1}{n} \cdot (x + 5) \right| = 0 < 1 \]

Series converges for all \( x \).

Radius of convergence is \( \infty \)

Interval of convergence is \( (-\infty, \infty) \).
9. (a) (5 points) Find the Maclaurin series for \( f(x) = \sin(x) \) by using the fact that \( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \) and \( \sin(x) = \int \cos(x) \, dx \).

\[
\sin(x) = \int \cos(x) \, dx
\]
\[
= \int \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right) \, dx
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{2n} \, dx
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{2n+1} x^{2n+1}
\]

(b) (5 points) Approximate the definite integral \( \int_{0}^{1} \frac{1}{x^3+1} \, dx \) using the 6th degree Taylor polynomial for \( \frac{1}{x^3+1} \) centered at 0. Hint: \( \frac{1}{x^3+1} = \sum_{n=0}^{\infty} (-1)^n x^{3n} \).

\[
\frac{1}{x^3+1} \approx 1 - x^3 + x^6
\]

\[
\int_{0}^{1} \frac{1}{x^3+1} \, dx = \int_{0}^{1} \left( 1 - x^3 + x^6 \right) \, dx
\]
\[
= \left[ x - \frac{1}{4} x^4 + \frac{1}{7} x^7 \right]_{0}^{1}
\]
\[
= 1 - \frac{1}{4} + \frac{1}{7} = \frac{25}{28}
\]
10. (5 points) Complete 2 iterations of Newton’s method to approximate a zero of the function \( f(x) = x^3 + 3x - 1 \). Use \( x_1 = 1 \) as an initial guess. Express your answer as a fraction.

\[
\begin{align*}
\frac{f(x)}{g'(x)} &= x^3 + 3x - 1, \quad g'(x) = 3x^2 + 3, \\
x_{n+1} &= x_n - \frac{f(x_n)}{g'(x_n)} = x_n - \frac{x_n^3 + 3x_n - 1}{3x_n^2 + 3}.
\end{align*}
\]

\[
x_1 = 1
\]

\[
x_2 = x_1 - \frac{f(x_1)}{g'(x_1)} = 1 - \frac{1^3 + 3(1) - 1}{3(1)^2 + 3} = 1 - \frac{3}{6} = \frac{1}{2}
\]

\[
x_3 = x_2 - \frac{f(x_2)}{g'(x_2)} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right) - 1}{3\left(\frac{1}{2}\right)^2 + 3}
\]

\[
= \frac{1}{2} - \frac{\frac{1}{8} + \frac{3}{2} - 1}{\frac{3}{4} + 3} = 2 - \frac{\frac{5}{8}}{\frac{15}{4}} = \frac{1}{3}
\]
11. (5 points) Consider an 8 by 8 checkerboard. Place 1 penny on square 1; place 2 pennies on square 2; place 4 pennies on square 3; place 8 pennies on square 4; place 16 pennies on square 5; and etc. How many pennies are needed to cover all the squares on the checkerboard?

\[ \text{Number of pennies} = \sum_{n=0}^{b^2} 2^n \]

Note \[ \sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r} \]

\[ S_0 \sum_{b^2} 2^n = \frac{(1-2^{64})}{(1-2)} \]
\[ = (1-2^{64}) \]
\[ = 2^{64} - 1 \]

12. (5 points) Find the function \( f(x, y) \) such that

\[ f_x = \sin(y) + y^2 e^x - 1, \quad f_y = x \cos(y) + 2ye^x + 2y, \text{ and } f(0, 0) = 0. \]

\[ f = \int f_x \, dx = \int \sin(y) + y^2 e^x - 1 \, dx = xy \sin(y) + ye^x - x + C'(y) \]

Where \( C'(y) \) is a function of \( y \).

\[ f = \int f_y \, dy = \int x \cos(y) + 2y e^x + 2y \, dy = xy \sin(y) + ye^x + y^2 + C''(x) \]

Where \( C''(x) \) is a function of \( x \).

So, \( f = xy \sin(y) + ye^x + y^2 - x + C \) where \( C \) is a constant.

**END OF EXAM**