MAT128A: Numerical Analysis
Lecture Fifteen: Chebyshev Interpolation, Again

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We first showed the existence of interpolating polynomials.

**Theorem**

If $x_0, x_1, \ldots, x_N$ are distinct points on the real line and $f : \mathbb{R} \rightarrow \mathbb{R}$, then there is a unique polynomial $p$ of degree $N$ which interpolates $f$ at the points $x_0, \ldots, x_N$.

Recall that $p$ interpolates $f$ at the nodes $x_0, \ldots, x_N$ means that

$$f(x_j) = p(x_j) \text{ for all } j = 0, 1, \ldots, N.$$
Next, we show that the truncated Chebyshev expansion for $f$ interpolates $f$ at the points of the Chebyshev grid.

**Theorem**

Suppose that $f : [-1, 1] \rightarrow \mathbb{R}$ is a continuous function, $x_0, x_1, \ldots, x_N$ are defined by

$$x_j = \cos \left( \frac{j + \frac{1}{2}}{N + 1} \pi \right),$$

and $a_0, a_1, \ldots, a_N$ are given by the formula

$$a_n = \frac{2}{N + 1} \sum_{j=0}^{N} f(x_j) T_n(x_j).$$

Then

$$\sum_{n=0}^{N} a_n T_n(x)$$

is the unique polynomial of degree $N$ which interpolates $f$ at the points $x_0, x_1, \ldots, x_N$. 

We then developed a constructive formula for the polynomial interpolating a function at any given set of nodes.

**Theorem**

*If* \( x_0, x_1, \ldots, x_N \) *are distinct real numbers, then*

\[
L(x) = \sum_{j=0}^{N} f(x_j) \prod_{0 \leq i \leq N, i \neq j} \frac{x - x_i}{x_j - x_i}
\]

*is the unique polynomial of degree* \( N \) *such that*

\[
L(x_i) = f(x_i) \quad \text{for all} \quad i = 0, 1, \ldots, N.
\]
Finally, we developed an expression for interpolation error.

**Theorem**

Suppose that $f : [a, b] \to \mathbb{R}$ is an element of $C^{N+1}[a, b]$, that

$$x_0 < x_1 < \ldots < x_N$$

are points in $[a, b]$, and that $p$ is the unique polynomial of degree $N$ which interpolates $f$ at the nodes $x_0, \ldots, x_N$. Then, for each $x \in [a, b]$, there is a point $\xi_x \in (a, b)$ such that

$$f(x) = p(x) + \frac{f^{(N+1)}(\xi_x)}{(N + 1)!} (x - x_0)(x - x_1) \cdots (x - x_N).$$
Good Interpolation Nodes

Now we will investigate the question of what interpolations nodes should be chosen.

An obvious strategy is to try to minimize the magnitude of the error term

\[ \frac{f^{(N+1)}(\xi_x)}{(N+1)!} (x - x_0)(x - x_1) \cdots (x - x_N). \]

We cannot hope to control the magnitude of the \((N + 1)^{st}\) derivative of \(f\) if we want to choose nodes which do not depend on what function we are interpolating, but we can choose nodes

\[ x_0, x_1, \ldots, x_N \]

which minimize the magnitude of

\[ (x - x_0)(x - x_1) \cdots (x - x_N). \]
Good Interpolation Nodes

Before we state the next theorem about “good interpolation node,” let’s recall a few facts.

We say that a polynomial is monic if its leading coefficient is 1.

The uniform norm of a function $f : [-1, 1] \to \mathbb{R}$ is

$$\sup_{-1 \leq x \leq 1} |f(x)| .$$

We denote it by $\| f \|_{\infty}$.

We also recall that the leading coefficient of $T_{n+1}$ is $2^n$ (this follows by induction and the recurrence relations).
Good Interpolation Nodes

Theorem

For each $j = 0, 1, \ldots, N$, let

$$x_j = \cos \left( \frac{j + \frac{1}{2}}{N + 1} \pi \right).$$

Then

$$(x - x_0)(x - x_1) \cdots (x - x_N) = \frac{1}{2^N} T_{N+1}(x)$$

is the monic polynomial of degree $N + 1$ with the smallest possible uniform norm, and that norm is $2^{-N}$. 
Proof:

First of all, let’s make sure we understand why

$$(x - x_0)(x - x_1) \cdots (x - x_N) = \frac{1}{2^N} T_{N+1}(x).$$

We know that $T_{N+1}$ is a polynomial of degree $N + 1$, and the formula

$$T_{N+1}(x) = \cos((N + 1) \arccos(x))$$

implies that its roots are

$$\cos \left( \frac{j + \frac{1}{2} \pi}{N + 1} \right) \quad j = 0, 1, \ldots, N + 1$$

since the zeros of cosine are

$$\frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}.$$
Since \( T_{N+1} \) and \((x - x_0)(x - x_1) \cdots (x - x_N)\) have the same roots, there must be a constant \( C \) such that

\[
T_{N+1}(x) = C(x - x_0)(x - x_1) \cdots (x - x_N).
\]

That the correct constant \( C \) is \( 2^{-N} \) then follows from the fact that the leading coefficient (i.e., the coefficient of \( x^{N+1} \)) of

\[
(x - x_0)(x - x_1) \cdots (x - x_N)
\]

is 1 while the leading coefficient of \( T_{N+1} \) is \( 2^N \).

So

\[
T_{N+1}(x) = 2^N (x - x_0)(x - x_1) \cdots (x - x_N).
\]

We will now show that \( 2^{-N} T_{N+1} \) is the monic polynomial of degree \( N + 1 \) with the smallest uniform norm. That it is a monic polynomial means that its leading coefficient is 1.
Suppose that $p$ is a monic polynomial of degree $N + 1$ such that

$$|p(x)| < 2^{-N}$$

for all $x \in [-1, 1]$. For each $j = 0, 1, \ldots, N, N + 1$, let

$$y_j = \cos \left( \frac{\pi}{N + 1} j \right).$$

These are the minima and maxima of the Chebysev polynomial $T_{N+1}$ and the value of

$$T_{N+1} \left( \cos \left( \frac{\pi}{N + 1} j \right) \right)$$

alternatives between 1 and $-1$. It follows that

$$p(y_0) < 2^{-N} T_{N+1}(y_0)$$
$$p(y_1) > 2^{-N} T_{N+1}(y_1)$$
$$p(y_2) < 2^{-N} T_{N+1}(y_2)$$
$$\vdots$$
We let
\[ q(x) = p(x) - 2^{-N} T_{N+1}(x). \]

Then \( q \) alternates signs between the points \( y_0, y_1, \ldots, y_N, y_{N+1} \), so it has at least \( N + 1 \) zeros. But \( q \) is a polynomial of degree at most \( N \) since the leading term in \( p \) and \( 2^{-N} T_{N+1} \) cancel. It follows that \( q \) must be identically zero (the only way a polynomial of degree less than or equal to \( N \) can have \( N + 1 \) zeros is if it is identically zero). In other words, we must have
\[ p(x) = 2^{-N} T_{N+1}(x). \]

But this contradicts our assumption that
\[ \lvert p(x) \rvert < 2^{-N}, \]

since \( 2^{-N} T_{nN1}(x) \) assumes the value \( 2^{-N} \). We conclude that there can be no monic polynomial \( p \) such that
\[ \lvert p(x) \rvert < 2^{-N} \]

for all \( x \in [-1, 1] \).
We conclude this theorem that Chebyshev nodes are reasonably good interpolation nodes.

Note, though, that this does not mean that the polynomial

\[
p(x) = \sum_{n=0}^{N} a_n T_n(x), \quad a_n = \frac{2}{N+1} \sum_{j=0}^{N} f \left( \cos \left( \frac{j + \frac{1}{2}}{N+1} \right) \right) T_n \left( \cos \left( \frac{j + \frac{1}{2}}{N+1} \right) \right)
\]

minimizes the error

\[
\{\|f - q\|_{\infty} : q \text{ is a polynomial of degree } N + 1\},
\]

only that it minimizes a factor which appears in one particular expression for the error in the Lagrange formula.
Minimax Approximations

**Theorem**

Suppose that \( f : [-1, 1] \to \mathbb{R} \) is a continuous function. There is a unique polynomial \( p^*_N \) of degree \( N \) such that

\[
\| f - p^*_N \|_{\infty} = \min \{ \| f - q \|_{\infty} : q \text{ is a polynomial of degree } N \},
\]

where \( \| \cdot \|_{\infty} \) is the uniform norm on \([-1, 1]\). We call \( p^*_N \) the minimax polynomial of degree \( N \) for the function \( f \).

The polynomial \( p^*_N \) is called the minimax polynomial because

\[
\| f - p^*_N \|_{\infty} = \min_{q \in \mathbb{P}^n} \max_{x \in [-1, 1]} |f(x) - q(x)|,
\]

where \( \mathbb{P}^n \) denotes the vector spaces of polynomials of degree less than or equal to \( N \).
Minimax Approximations vs Chebyshev Approximations

Computing the minimax polynomials is computationally difficult, and there is very little profit in it, as the next theorem demonstrates.

**Theorem**

Suppose that \( f : [-1, 1] \to \mathbb{R} \) is a continuous function, that \( p_N^* \) is the minimax polynomial of degree \( N \) for \( f \), that \( \{a_n\} \) are the Chebyshev coefficients of \( f \) — that is,

\[
a_n = \frac{2}{\pi} \int_0^\pi f(x) T_n(x) \frac{dx}{\sqrt{1 - x^2}}
\]

— and that

\[
p_N(x) = \sum_{n=0}^N a_n T_n(x).
\]

Then

\[
\|f - p_N\|_\infty \leq \left( 4 + \frac{4}{\pi^2} \log(N) \right) \|f - p_N^*\|_\infty
\]

and

\[
\frac{\pi}{4} |a_{N+1}| \leq \|f - p_N^*\|_\infty.
\]
Minimax Approximations

We will not prove the preceding theorem, but we will discuss some of its implications.

We note first that the theorem bounds the error in the approximation of $f$ by the truncated Chebyshev expansion with exact coefficients. This is not a serious difficulty, though, because we know that if

$$
\tilde{P}_N(x) = \sum_{n=0}^{N} \tilde{a}_n T_n(x) \quad \text{with} \quad \tilde{a}_n = \frac{2}{N+1} \sum_{j=0}^{N} f \left( \cos \left( \frac{j + \frac{1}{2}}{N} \pi \right) \right) T_n \left( \cos \left( \frac{j + \frac{1}{2}}{N} \pi \right) \right),
$$

then

$$
\left\| P_N(x) - \tilde{P}_N(x) \right\|_\infty \leq \sum_{n=N+1}^{\infty} |a_n|.
$$

This means that if the Chebyshev coefficients of $f$ decay rapidly, then

$$
P_N(x) \approx \tilde{P}_N(x)
$$

once $N$ is of moderate size.
The logarithm is a very slowly growing function:

This means that unless $N$ is very large, the inequality

$$\|f - p_N\|_\infty \leq \left(4 + \frac{4}{\pi^2} \log(N)\right) \|f - p^*_N\|_\infty$$

shows that the minimax approximation of the continuous function $f$ is not that much better than the Chebyshev approximation.
Minimax Approximations vs Chebyshev Approximations

Moreover, the second bound

\[ \frac{\pi}{4} |a_{N+1}| \leq \| f - p_N^* \|_\infty \]

is useful for showing that if the Chebyshev coefficients of a function decay rapidly, then the minimax approximation is not much better than the Chebyshev approximation.

For instance, suppose that \( |a_n| \leq r^{-n} \). Then

\[ \| f - p_N \|_\infty \leq \sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} r^{-n} = \frac{r^{-N}}{1 - r} = \frac{r}{1 - r} |a_{N+1}| \leq \frac{4r}{\pi(1 - r)} \| f - p_N^* \|_\infty. \]

This bound can be improved, but in this form it already shows that if \( f \) is analytic then accuracy of the Chebyshev approximation of \( f \) is within a constant factor of the accuracy of the minimax approximation.

The same can be shown to be true if \( f \) is \( C^k \), although doing so requires a much more involved argument.