MAT168
TAKE-HOME FINAL
WEDNESDAY JUNE 4, 2008

DUE WEDNESDAY JUNE 11, 2008

INSTRUCTIONS:

1. You are welcome to consult your notes from class or the book, but refrain from using other sources.

2. Answer each question on a separate sheet of paper; be sure to label the paper with the number of problem and please put your answers in the same order as the questions below. If you can’t answer a question, then leave your solution blank.

3. Turn your exam into my box in the math department mail room (in MSB). Please note that the mail room is only accessible from 9:00 AM to 12:10 PM and 1:00 PM to 4:00 PM Monday through Friday.

4. **Late exams will not be accepted!!!!** Turn it whatever you have finished by June 11, 2008.

5. You may use a computer to solve any linear program on this exam. You might wish to use MATLAB’s linprog command, your own project, or the primitive pivot_step.m routine (or, for that matter, any other method you would like).

**Good Luck!**
Problem 1. Find a $4 \times 2$ matrix $A$, a vector $b$ of length 2, and a vector $c$ of length 2 for which the linear programs

$$\begin{align*}
\text{max:} & \quad c^t x \\
\text{subject to:} & \quad Ax \leq b \\
& \quad 0 \leq x
\end{align*}$$

(1)

and

$$\begin{align*}
\text{max:} & \quad c^t x \\
\text{subject to:} & \quad Ax \leq b \\
& \quad x \text{ free}
\end{align*}$$

(2)

are both bounded and feasible AND, for which the solution of (2) is strictly greater than the solution of (1).

(a) Sketch a picture of the feasible region of (1) and indicate a vertex on the boundary at which the problem (1) has a solution.

(b) Sketch a picture of the feasible region of (2) and indicate a vertex on the boundary at which the problem (2) has a solution.

(c) Is it possible for the solution of (1) to be larger than the solution of (2)? Explain why or why not.

Note: this problem does not require computation at all if you are clever about picking $A$.

Problem 2. Consider a linear program of the form

$$\begin{align*}
\text{max:} & \quad c^t x \\
\text{subject to:} & \quad Ax = b \\
& \quad 0 \leq x \leq u
\end{align*}$$

(3)

where $A$ is an $m \times n$ matrix of rank $m$, $c$ and $b$ are arbitrary vectors of length $n$, and $u$ is a vector of length $n$ with nonnegative entries.

(a) Show that the dual of a problem of the form (3) is always feasible.

(b) Give an example of an $3 \times 6$ matrix $A$ of rank $m$ and a vector $b \in \mathbb{R}^m$ such that the linear program (3) is infeasible for any choice of $u$.

(c) Show that if the program (3) is infeasible, then its dual is unbounded.

Problem 3. Given an $m \times n$ matrix $A$, we say that a vector

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$$

is a minimum $l^1$ norm solution of $Ax = b$ if $Ax^* = b$ and

$$|x_1^*| + |x_2^*| + \ldots + |x_n^*| \leq |x_1| + |x_2| + \ldots + |x_n|$$

for all solutions $x$ of $Ax = b$.

(a) Write down a linear program for finding a minimum $l_1$ norm solution of a linear system of equations $Ax = b$ where $A$ is an $m \times n$ matrix.

(b) Now let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & -1 & 3 \\ 1 & 1 & -1 & 3 & -6 \end{pmatrix}$$
and let \( b \) be the vector
\[
b = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}.
\]

Find a minimum \( l^1 \) norm solution of the equation \( Ax = b \) using the simplex method. That is, write down an optimal feasible vector for the linear program.

Problem 4. (a) Show that Newton's Method can be used to approximate \( \frac{1}{b} \), where \( b \) is a positive constant, without performing any division operations. (Hint: consider the function \( f(x) = 1/(xb) - 1 \).)

(b) Use Newton's Method to approximate \((2)^{1/3}\) (the real-valued third root of 2). Start with the initial guess of \( x_0 = 6/5 \) and compute three iterates \( x_1, x_2, \) and \( x_3 \).

All of the computations for this part must be done in EXACT arithmetic (i.e., not using approximate computer arithmetic). Write each iterate as a fraction.

Problem 5. The diagram in Figure 1 defines a minimum-cost network flow problem.

(a) Write down the linear program for this network flow problem.

(b) Find a solution (i.e., an optimal basic feasible vector) with integer entries for the program using the simplex method. A solution consists of a flow value for each arc in the network; indicate clearly the value of your solution at each arc in the network.

![Figure 1](image_url)

**Figure 1.** The numbers above the nodes are supplies (negative values represent demands) and the numbers shown above arcs are shipping costs.