Problem 1. (20 pts) Let \( A \) be the \( 4 \times 3 \) matrix

\[
A = \begin{pmatrix}
1 & 2 & 1 \\
-1 & 2 & 2 \\
3 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

(a) Find a basis for the column space of \( A \).
(b) Find a basis for the null space of \( A \).
(c) Find a basis for the row space of \( A \).

Solution. The matrix \( A \) is row equivalent to

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

which is in RREF. We see from this that

(a) the three columns of \( A \) are a basis for the column space of \( A \),
(b) The null space of \( A \) is \( \{0\} \)
(c) The vectors

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

form a basis for the row space of \( A \). Alternately, the first three rows of \( A \) form a basis for the row space of \( A \). Note, however, that the last three do not.
Problem 2. (15 pts) Consider the polynomials
\[ p_1(t) = 1 + t \]
\[ p_2(t) = 1 + 2t \]
\[ p_3(t) = 1 - t + 2t^2 + t^3 \]
\[ p_4(t) = 1 - t + t^2 + t^3. \]
Show that \( \{ p_1, p_2, p_3, p_4 \} \) is a basis for the vector space of polynomials of degree less than or equal to 3.

Solution. First we form the matrix \( P \) of the coefficients of the polynomials \( p_1, \ldots, p_4 \) with respect to the canonical basis \( \{1, t, t^2, t^3\} \) for the vector space of polynomials of degree \( \leq 3 \):
\[
P = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & -1 & -1 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]
The polynomials \( p_1, p_2, p_3, p_4 \) form a basis if and only if this matrix can be row reduced to the identity, so this problem can be solved by row reducing the matrix \( P \).

Alternately, one can compute the determinant of \( P \) (recall that the determinant of a \( n \times n \) matrix is nonzero if and only if it can be row reduced to the identity). We compute:
\[
\det P = \begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & -1 & -1 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 1
\end{vmatrix} = \begin{vmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & -1 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & 1 & 0
\end{vmatrix} = \begin{vmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{vmatrix} = \begin{vmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{vmatrix} = 1,
\]
where the last step follows since the determinant of an upper triangular matrix is the product of its diagonal entries. It follows that the polynomials \( p_1, \ldots, p_4 \) form a basis.
Problem 3. (15 pts) Let

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

(a) What is the rank of \( A \)?
(b) What is the dimension of the null space of \( A \)?
(c) What is the dimension of the row space of \( A \)?
(d) What is the rank of \( B \)?
(e) What is the dimension of the null space of \( B \)?
(f) What is the dimension of the row space of \( B \)?

Solution. Since both \( A \) and \( B \) are already in RREF, we can answer each of the above questions very easily.
(a) The rank of a matrix is the dimension of its column space (which is also equal to the dimension of its row space). Since \( A \) has two leading ones, the answer is 2.
(b) The dimension of the null space of \( A \) is the number of columns of \( A \) minus its rank, which in this case is 1.
(c) The dimension of the row space of \( A \) is also equal to the number of leading ones, which is again 2.
(d) 3
(e) 0
(f) 3
Problem 4. (10 pts) Which of the following subsets of $\mathbb{R}^3$ are subspaces?

\[ V_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y = 0 \right\} \]

\[ V_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = y = 0 \right\} \]

\[ V_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 2 \right\} \]

Solution. $V_1$ and $V_2$ are subspaces of $\mathbb{R}^3$ and $V_3$ is not a subspace. The question did not ask for us to justify our answer, but we will do so here anyway.

The subspaces $V_1$ and $V_2$ are both null spaces of matrices; for instance, $V_1$, is the null space

\[ \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \left| \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right. \right\}. \]

One the other hand, it is easy to see that the vector

\[ v = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \]

is in the set $V_3$ while its scalar multiple $2v$ is not.
Problem 5. (20 pts) Let $T$ be the mapping $\mathbb{R}^3 \to \mathbb{R}^1$ defined by $Tv = \|v\|$ where $\|v\|$ is the length of the vector $v$; i.e.,

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Show that $T$ is not a linear transformation.

Solution. We must produce a counterexample to show that $T$ does not satisfy one or both of the two properties required of linear transformations.

Let $v$ be the vector

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and let $\lambda$ be the scale $\lambda = -1$. Then

$$T(v) = 1$$

but

$$T(\lambda v) = 1 \neq -T(v) = \lambda T(v).$$
Problem 6. (10 pts) Suppose that $A$ is an $3 \times 3$ matrix of rank 3. Show that the system of equations $Ax = 0$ has only the trivial solution $x = 0$.

Solution. The rank of a $m \times n$ matrix plus the dimension of its null space is equal $n$. So, since $A$ has 3 columns, we know that

$$3 = \text{rank}(A) + \dim \text{null}(A).$$

Since we are told that rank$(A)$ is 3, it follows that the dimension of the null space of $A$ is 0. That is,

$$\text{null}(A) = \{0\}.$$  

In other words, the only solution to $Ax = 0$ is the trivial solution $x = 0$. 
Problem 7. (10 pts) What is the angle between the vectors

\[ v = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \]

and

\[ w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]?

Solution. We use the formula

\[ \cos(\theta) = \frac{v \cdot w}{\|v\|\|w\|} \]

where \( \theta \) is the angle between \( v \) and \( w \). Since

\[ v \cdot w = 1 \cdot 1 + (-2) \cdot 1 + 1 \cdot 1 = 0, \]

it follows that the angle \( \theta \) between \( v \) and \( w \) is \( \pi/2 \) (or ninety degrees).