Problem 1. (10 pts) Let $A$ be the matrix

$$
\begin{pmatrix}
1 & 1 & 1 \\
2 & 1 & 1 \\
2 & -2 & 2
\end{pmatrix}.
$$

(a) Find the inverse of $A^{-1}$ of $A$.

(b) Use the inverse of $A$ to find $x$ such that

$$Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Solution. The inverse of the matrix $A$, which we can calculate by row reducing the augmented matrix $(A | I)$, is

$$A^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1/2 & 0 & -1/4 \\ 3/2 & -1 & 1/4 \end{pmatrix}.$$

To answer part (b), we form the product

$$x = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/4 \\ 1/4 \end{pmatrix}.$$
**Problem 2.** (15 pts) Let $A$ be the matrix

$$
A = \begin{pmatrix}
1 & 1 & 2 & 2 & 2 \\
2 & 2 & 1 & 5 & 1 \\
-1 & -1 & 1 & 3 & 1 \\
-1 & -1 & 4 & -4 & 4
\end{pmatrix}
$$

(a) Find a basis for the null space of $A$.
(b) Find a basis for the column space of $A$.
(c) Find a basis for the row space of $A$.
(d) What is the rank of $A$?

**Solution.** The matrix $A$ row reduces (without any rows being exchanged) to

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

From this, we see that the first, third, and fourth columns

$$
\begin{pmatrix}
1 \\
2 \\
-1 \\
-1
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
1 \\
4
\end{pmatrix}, \quad \text{and} \quad
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

form a basis for the column space of $A$. We also see that the first three rows of $A$ form a basis for the row space of $A$. Choosing $x_2$ and $x_5$ are free variables leads us to the basis

$$
\left\{ \begin{pmatrix}
-1 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} \right\}
$$

for the null space of $A$. The rank of $A$ is, of course, the dimension of the column space and so the rank of $A$ is 3.
Problem 3. (15 pts) Suppose that $A$ is a $3 \times 3$ matrix such that

$$A \cdot v_1 = v_1 - v_2$$
$$A \cdot v_2 = v_2 - v_3$$
$$A \cdot v_3 = v_3,$$

where

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) What is $A \cdot x$ where

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) Compute $\det(A)$.

(c) What is the characteristic polynomial of the matrix $A$?

Solution. In order to answer (a), we must find the coefficients of the vector $x$ in the basis $v_1, v_2, v_3$. We can do so by row reducing the augmented matrix

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

We find that $x = v_1 + v_2 + v_3$. This allows us to compute $Ax$:

$$Ax = A v_1 + A v_2 + A v_3 = (v_1 - v_2) + (v_2 - v_3) + v_3 = v_1.$$

To answer (b) and (c), we form the matrix of $A$ with respect to the basis $v_1, v_2, v_3$, which is:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

We know that the matrix $B$ is similar to $A$ and so $A$ and $B$ have the same determinant and characteristic polynomial. So that determinant of $A$ is

$$\det(A) = \det(B) = 1$$

and the characteristic polynomial is

$$\det(A - \lambda I) = \det(B - \lambda I) = (1 - \lambda)^3.$$
Problem 4. (15 pts) Let $A$ be the matrix
\[
\begin{pmatrix}
7 & -1 & 2 \\
-1 & 7 & -2 \\
2 & -2 & 10
\end{pmatrix}.
\]

(a) What is the characteristic polynomial of $A$?

(b) Find all of the eigenvalues of $A$.

(c) For each eigenvalue $\lambda$ of $A$ find a basis for the vector space
$V_\lambda = \{v : Av = \lambda v\}$.

Solution. (a) The characteristic polynomial of $A$ is
\[
\det(A - \lambda I) = \begin{vmatrix}
7 - \lambda & -1 & 2 \\
-1 & 7 - \lambda & -2 \\
2 & -2 & 10 - \lambda
\end{vmatrix} \\
= \begin{vmatrix}
6 - \lambda & -1 & 2 \\
6 - \lambda & 7 - \lambda & -2 \\
0 & -2 & 10 - \lambda
\end{vmatrix} \\
= \begin{vmatrix}
6 - \lambda & -1 & 2 \\
0 & 8 - \lambda & -4 \\
0 & -2 & 10 - \lambda
\end{vmatrix} \\
= (6 - \lambda)((8 - \lambda)(10 - \lambda) - 8) \\
= -(\lambda - 6)(\lambda - 6)(\lambda - 12).
\]

(b) The eigenvalues of $A$ are $\lambda = 12$ and $\lambda = 6$.

(c) We find a basis for $V_\lambda$ by finding a basis for the null space of $A - \lambda I$. Since
\[
A - 6I = \begin{pmatrix}
1 & -1 & 2 \\
-1 & 1 & -2 \\
2 & -2 & 4
\end{pmatrix} \sim \begin{pmatrix}
1 & -1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
the vectors
\[
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
-2 \\
0 \\
1
\end{pmatrix}
\]
form a basis for $V_6$. A similar computation shows that
\[
\begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix}
\]
is a basis for $V_{12}$. 
Problem 5. (15 pts) As usual, \( \mathbb{P}^n \) denotes the vector space of all polynomials of degree less than or equal to \( n \). Let \( L \) be the linear transformation \( \mathbb{P}^3 \to \mathbb{P}^4 \) defined by the formula

\[
L(p(t)) = p(0) + tp(t) + t^2 p'(t)
\]

where \( p'(t) \) denotes the derivative of \( p(t) \) with respect to \( t \).

Find the matrix of \( L \) with respect to the bases

\[
S = \{1, 1 + t, 1 + t^2, 1 + t^3\}
\]

and

\[
T = \{1, 1 + t, 1 + t^2, 1 + t^3, 1 + t^4\}.
\]

Note: \( S \) is a basis for \( \mathbb{P}^3 \) and \( T \) is a basis for \( \mathbb{P}^4 \), so the “input basis” for \( L \) is \( S \) and the “output basis” is \( T \).

Solution. We first compute \( L(s_j) \) for each vector \( s_j \) in \( S \):

\[
L(1) = 1 + t \\
L(1 + t) = 1 + t(1 + t) + t^2(1) = 1 + t + 2t^2 \\
L(1 + t^2) = 1 + t(1 + t^2) + t^2(2t) = 1 + t + 3t^3 \\
L(1 + t^3) = 1 + t(1 + t^3) + t^2(3t^2) = 1 + t + 4t^4.
\]

Now we must write each of the vectors \( L(s_j) \) with respect to the basis \( T \). We do that by row reducing the augmented matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & -3 & -4 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 2 & -3 & -4 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 4
\end{pmatrix}.
\]

So the matrix of \( L \) with respect to \( S \) and \( T \) is

\[
\begin{pmatrix}
0 & 2 & -3 & -4 \\
1 & 1 & 1 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}.
\]
Problem 6. (10 pts) Suppose that \(\{v_1, \ldots, v_n\}\) and \(\{w_1, \ldots, w_n\}\) are vectors in \(\mathbb{R}^n\) such that

\[
v_i \cdot w_j = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j,
\end{cases}
\]

where \(v_i \cdot w_j\) is the inner product of \(v_i\) and \(w_j\). Show that the set \(\{v_1, \ldots, v_n\}\) is linearly independent.

Solution. In order to show that \(\{v_1, \ldots, v_n\}\) is linearly independent, we must show that

\[
\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0,
\]

then \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0\). To do that we take the inner product of (1) with \(w_j\); that is, suppose

\[
\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.
\]

Then

\[
w_j \cdot (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = 0.
\]

But

\[
w_j \cdot (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = \alpha_1 (w_j \cdot v_1) + \ldots + \alpha_j (w_j \cdot v_j) + \ldots \alpha_n (w_n \cdot v_n)
\]

\[
= \alpha_j.
\]

And so (1) implies that \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0\).
**Problem 7.** (10 pts) Find orthonormal bases for the column space and null space of the matrix

\[ A = \begin{pmatrix} 2 & -2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -2 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{pmatrix}. \]

**Solution.** We first row reduce the matrix \( A \) in order to find a bases for the column space and null space of \( A \). The matrix \( A \) is row equivalent to

\[ \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

From this we see that

\[ \{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \} \]

is a basis for the column space of \( A \) and

\[ \{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \} \]

is a basis for the null space. In order to find orthonormal bases for these subspaces, we will use the Gram-Schmidt algorithm to orthonormalize these bases. To form an orthonormal basis for the column space, we first set

\[ \tilde{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \]

and then let

\[ v_1 = \tilde{v}_1/\|\tilde{v}_1\| = \tilde{v}_1/\sqrt{10}. \]

Then we let

\[ \tilde{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot v_1 \]

\[ = \frac{2}{5} \begin{pmatrix} 1 \\ -2 \\ 1 \\ -2 \end{pmatrix}. \]

Finally, we set

\[ v_2 = \tilde{v}_2/\|\tilde{v}_2\| = \frac{\sqrt{10}}{4} \tilde{v}_2. \]
So

\[
\begin{bmatrix}
\frac{1}{\sqrt{10}} & 2 \\
1 & 1 \\
2 & 1
\end{bmatrix}, \quad \frac{1}{\sqrt{10}} \begin{bmatrix}
1 \\
-2 \\
1 \\
-2
\end{bmatrix}
\]

is an orthonormal basis for the column space of $A$. The same procedure applied to the basis for the null space will give us an orthonormal basis for the null space of $A$. 
Problem 8. (10 pts) Consider the linear transformation $L : \mathbb{P}^2 \to \mathbb{R}^3$ defined by

$$L(a + bt + ct^2) = \begin{pmatrix} a - b + c \\ b + c \\ a - b - c \end{pmatrix}.$$ 

(a) Find a basis for the kernel of $L$.
(b) Find the rank of $L$.
(c) Is $L$ one-to-one?
(d) Is $L$ onto?

Solution. We first find a matrix $A$ for the linear transformation $L$ with respect to the canonical bases $\{1, t, t^2\}$ for $\mathbb{P}^2$ and $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$. That matrix is

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$ 

We can now answer all of the questions (a)-(d) by row reducing the matrix $A$. We see that $A$ is row equivalent to the identity matrix, and so:
(a) The kernel of $A$ is $\{0\}$.
(b) The rank of $L$ is 3.
(c) $L$ is indeed one-to-one (since the kernel is $\{0\}$).
(d) $L$ is also onto.