1. Please write your name on the cover sheet.
2. Write your answers on the exam paper.
3. Calculators, cell phones, and other electronic devices are not allowed.
4. Good luck!
Problem 1. Let

\[
A = \begin{pmatrix}
2 & 2 & 1 & 4 \\
1 & 1 & 2 & 2 \\
1 & 1 & 1 & 2 \\
1 & 1 & -1 & 2
\end{pmatrix}
\]
and suppose that \( B \) is a \( 4 \times 6 \) matrix which row reduces to

\[
\begin{pmatrix}
1 & 2 & 0 & 1 & 2 & -1 \\
0 & 0 & 1 & 2 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(a) Find a basis for the column space of \( A \).

(b) Find a basis for the null space of \( A \).

(c) What is the dimension of the null space of \( B \)?

(d) Use the fact that

\[
B \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}
\]

to find the set of all of the solutions of the equation

\[
B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}
\]

We begin by row reducing the matrix \( A \) to

\[
\begin{pmatrix}
1 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
(a) Since the RREF of $A$ has pivots in the first and third column,
\[
\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix} \right\}
\]
is a basis for the column space of $A$.

(b) We find a basis for the null space by solving the system $Ax = 0$. In particular, we can see from the RREF of $A$ that all solutions of this equation are of the form
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2t \\ 0 \\ 0 \\ t \end{pmatrix}.
\]
So
\[
\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
is a basis for the null space of $A$.

(c) The matrix $B$ row reduces to a matrix in RREF with 2 pivots, so the dimension of the column space of $B$ is 2. Since there are 6 columns in $B$, it follows that the dimension of the null space is 4.

(d) The nullspace of $B$ is spanned by
\[
\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}
\]
which means that any solution of the given equation is of the form
\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -2 \\ 0 \\ -3 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.
\]
Problem 2. Suppose that $A$ is a $3 \times 3$ matrix such that

$$Av_1 = v_1 - v_3$$
$$Av_2 = v_2 - v_3$$
$$Av_3 = 2v_1 - 2v_3$$

where

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Find $Av$ where $v$ is the vector

$$v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) What is the determinant of $A$?

We begin by writing the matrix $v$ in terms of the basis $\{v_1, v_2, v_3\}$. You can do this by solving a linear system, but it is easy to see that

$$v = v_1 - v_3$$

so that

$$Av = A(v_1 - v_3)$$
$$= Av_1 - Av_3$$
$$= (v_1 - v_3) - (2v_1 - 2v_3)$$
$$= -v_1 + v_3$$
$$= \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Note that the whole point of the part (a) is that you do not need to explicitly know what the matrix $A$ is, only its action on a basis.

Part (b) is meant to be tricky. What I have specified is the matrix of the linear transformation

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

with respect to the basis $\alpha = \{v_1, v_2, v_3\}$. In particular,

$$[T]^{\alpha}_\alpha = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & -1 & -2 \end{pmatrix}.$$
The observation necessary to solve part (b) is that the determinant of $A$ is the same as the determinant of this matrix. That is because

$$A = B^{-1}[T]_{\alpha}^\alpha B$$

where $B$ is the change of basis matrix

$$B = [I]_{\alpha}^{\text{std}}.$$ 

Here, std refers to the standard basis. So we have

$$\det(A) = \det(B^{-1}[T]_{\alpha}^\alpha B) = \det(B^{-1}) \det(T_{\alpha}) \det(B) = \det(B)^{-1} \det(B) \det(T_{\alpha}) = \det(T_{\alpha}).$$

Thus we can just compute the determinant of the matrix $[T]_{\alpha}^\alpha$ which is 0 since the third column is a multiple of the first column.
**Problem 3.** Compute the determinants of the matrices 

\[
A = \begin{pmatrix} 
1 & 2 & 0 & 0 \\
0 & 2 & 1 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 2 & 1 \\
\end{pmatrix} \quad \text{and} \quad 
B = \begin{pmatrix} 
1 & 2 & 0 & 2 \\
1 & 2 & 1 & 2 \\
1 & 0 & 1 & 2 \\
1 & 0 & 2 & 1 \\
\end{pmatrix}.
\]

We can use a single row operation to reduce the problem of computing the first determinant to that of computing the determinant of an upper triangle matrix:

\[
\begin{vmatrix} 
1 & 2 & 0 & 0 \\
0 & 2 & 1 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 2 & 1 \\
\end{vmatrix} = 
\begin{vmatrix} 
1 & 2 & 0 & 0 \\
0 & 2 & 1 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 \\
\end{vmatrix} = 1 \cdot 2 \cdot (-1) = -2.
\]

We use one column operation to reduce the second computation to the problem of finding a 3 × 3 determinant:

\[
\begin{vmatrix} 
1 & 2 & 0 & 2 \\
1 & 2 & 1 & 2 \\
1 & 0 & 1 & 2 \\
1 & 0 & 2 & 1 \\
\end{vmatrix} = 
\begin{vmatrix} 
1 & 2 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 2 & -1 \\
\end{vmatrix} = - 
\begin{vmatrix} 
1 & 2 & 0 \\
1 & 2 & 1 \\
1 & 0 & 1 \\
1 & 0 & 2 \\
\end{vmatrix} = -2. 
\]

\[
\begin{vmatrix} 
1 & 2 & 0 \\
0 & 2 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{vmatrix} = -2.
\]
Problem 4. Which of the following of these mappings are linear transformations and which are not? Justify each of your answers.

(a) The mapping $T : \mathbb{R}^3 \to \mathbb{R}$ defined by
$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{x^2 + y^2 + z^2}.$$ 

(b) The mapping $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by
$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} w \\ x \end{pmatrix}.$$

(c) The mapping $T : M_2 \to \mathbb{R}$ defined by $T(A) = \det(A)$. Here, $M_2$ is the vector space of $2 \times 2$ matrices with real entries and det refers to the determinant.

(a) The first transformation is not linear. This can be seen easily:
$$T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + T \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$ but
$$T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + T \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 + 1 = 2.$$

(b) The second transformation is linear. One way to see this is to observe that
$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$ 

(c) The third transformation is not linear since $\det(A + B)$ is not necessarily equal to $\det(A) + \det(B)$. 
Problem 5. Let $\mathbb{P}^3$ denote the vector space of polynomials of degree less than or equal to 3 and let $T$ be the linear transformation $T : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ defined by

$$T(p(x)) = xp'(x) + p(0).$$

Find the matrix of $T$ with respect to the basis

$$\alpha = \{1, 1 + x, 1 + x^2, 1 + x^3\}.$$

That is, find $[T]_\alpha^\alpha$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_4$ be the vectors in the basis $\alpha$ ordered as above. We first observe that

$$T(\alpha_1) = 1$$
$$T(\alpha_2) = x + 1$$
$$T(\alpha_3) = 2x^2 + 1$$
$$T(\alpha_4) = 3x^3 + 1.$$

In order to form the desired matrix we need to write each of these polynomials in terms of the output basis, which happens to be $\alpha$. To do this, we row reduce the augmented matrix

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 3
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & -1 & -2 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 3
\end{pmatrix}$$

Doing this gives the coefficients of the $T(\alpha_j)$ in terms of the basis $\alpha$. The final answer is

$$[T]_\alpha^\alpha = \begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{pmatrix}.$$