1. Please write your name on the cover sheet.
2. Write your answers on the exam paper.
3. Calculators, cell phones, and other electronic devices are not allowed.
4. Good luck!
1. Write down the augmented matrix for the linear system

\[
\begin{align*}
    x + 2y + z &= 5 \\
    x + 8y + z &= 11 \\
    x - 2y + z &= 1
\end{align*}
\]

Clearly state whether the system has no solutions, one solution, or an infinite number of solutions. Write down the set of all solutions.

The augmented matrix is

\[
\begin{pmatrix}
    1 & 2 & 1 & | & 5 \\
    1 & 8 & 1 & | & 11 \\
    1 & -2 & 1 & | & 1
\end{pmatrix}
\]

The row reduced echelon form for this matrix is

\[
\begin{pmatrix}
    1 & 0 & 1 & | & 3 \\
    0 & 1 & 0 & | & 1 \\
    0 & 0 & 0 & | & 0
\end{pmatrix}
\]

The set of solutions can be written as

\[
\begin{align*}
    x &= 3 - z \\
    y &= 1
\end{align*}
\]

with \( z \) a free variable.
2. For which values of $c$ does the linear system

\begin{align*}
y + z &= -1 \\
2x + 2y + z &= 1 \\
2x + 3y + 2z &= c
\end{align*}

have a solution?

We can solve this problem by forming the augmented matrix

$$
\begin{pmatrix}
0 & 1 & 1 & -1 \\
2 & 2 & 1 & 1 \\
2 & 3 & 2 & c
\end{pmatrix}
$$

and row reducing it to

$$
\begin{pmatrix}
1 & 0 & -1/2 & 3/2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & c
\end{pmatrix}.
$$

We see that the system can only have a solution if $c = 0$. 

3. Let $A$ and $B$ be the matrices

\[ A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \]

and

\[ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \]

(1) Is $A$ in row reduced echelon form?

(2) Is $B$ in row reduced echelon form?

(3) Write down all of the solutions of the equation $Ax = c$ where

\[ c = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \]

(4) Write down all of the solutions of the equation $Bx = d$ where

\[ d = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \]

(1) No.

(2) Yes.

(3) Since the matrix $A$ is upper triangular, we can solve the system $Ax = c$ using back subsitution. We see that

\[ x = \begin{pmatrix} -2/3 \\ 4/3 \\ -1/3 \end{pmatrix} \]

(4) Since the matrix $B$ is already in row reduced echelon form, it is easy to form the solutions for the system:
\[ x_1 = 2 \]
\[ x_2 = 1 - 2x_3 \]

with \( x_3 \) arbitrary.
4. Let

\( B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} \)

and suppose that \( A \) is a \( 3 \times 3 \) matrix such that

\( BA = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \).

Solve the linear system

\( A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}. \)

We can solve this problem by multiplying both sides of the equation (5) on the left by the matrix \( B \). This gives us

\( BA \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \)

or

\( \begin{pmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 8 \end{pmatrix}. \)

Since the matrix \( BA \) is lower triangular, we can solve this linear system using forward substitution:

\( x = -2 \)
\( y = -2 \)
\( z = 10/3 \)
5. Let $A$ and $B$ be the same $3 \times 3$ matrices that appeared in the last problem. Find the inverse of the matrix $B$ and use it to compute the matrix $A$.

We can row reduce the augmented matrix

$$
(6) \begin{pmatrix} 1 & 1 & -1 & | & 1 & 0 & 0 \\ 2 & 0 & 1 & | & 0 & 1 & 0 \\ 1 & -1 & 3 & | & 0 & 0 & 1 \end{pmatrix}
$$

to find the inverse of $B$. The matrix reduces to

$$
(7) \begin{pmatrix} 1 & 0 & 0 & | & -1/2 & 1 & -1/2 \\ 0 & 1 & 0 & | & 5/2 & -2 & 3/2 \\ 0 & 0 & 1 & | & 1 & -1 & 1 \end{pmatrix}
$$

so the inverse of $B$ is

$$
\begin{pmatrix} -1/2 & 1 & -1/2 \\ 5/2 & -2 & 3/2 \\ 1 & -1 & 1 \end{pmatrix}.
$$

And now we compute:

$$
A = B^{-1}BA
$$

$$
= \begin{pmatrix} -1/2 & 1 & -1/2 \\ 5/2 & -2 & 3/2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}
$$

$$
= \begin{pmatrix} 3 & 0 & -3/2 \\ -7 & 1 & 9/2 \\ -3 & 1 & 3 \end{pmatrix}
$$