MAT21C Practice Final
Sections C01-C05

WRITE YOUR NAME AND SECTION NUMBER ON THE FRONT COVER OF THE EXAM RIGHT NOW.

This is a closed note exam. No books, notes or other references are allowed.

No calculators are allowed.

No cell phones or other electronic devices are allowed.

Please write your answers directly on this exam.

Name: ___________________________ ___________________________

Section: ___________________________ ___________________________
**Problem 1.** (20 points) Determine whether each of the series below converges or diverges. Justify your answers.

(a) \[ \sum_{n=1}^{\infty} \frac{\log^2(n)}{n^2} \]

(b) \[ \sum_{n=2}^{\infty} \frac{\cos(n)}{n \log^2(n)} \]

(c) \[ \sum_{n=2}^{\infty} \frac{1}{n \log(n)} \]

(d) \[ \sum_{n=1}^{\infty} a_n, \]

where \( \{a_n\} \) is defined recursively by \( a_1 = 2 \) and \( a_{n+1} = 0.9a_n \) for \( n > 1 \).

**Outline of solution:** The series (a) converges by the limit comparison test; compare it with the series \( \sum n^{-3/2} \), for instance.

The series (b) converges absolutely and hence is convergent. This can be shown using the integral test and the substitution \( u = \log(x) \).

The series (c) diverges. This can be verified with the integral test.

The series (d) converges by the ratio test since \( a_{n+1}/a_n = 0.9 < 1 \).
Problem 2. (20 points) Find an equation for the plane tangent to the surface
\[-x^2 + y^2 + z^2 = 0\]
at the point \((1, 1, 0)\).

Solution: The gradient of the function \(g(x, y, z) = -x^2 + y^2 + z^2\) is
\[
\nabla g = \begin{pmatrix} -2x \\ 2y \\ 2z \end{pmatrix}.
\]
The vector \(\mathbf{N} = \nabla g(1, 1, 0) = (-2, 2, 0)\) is normal to the tangent plane at the point \((1, 1, 0)\). So the equation of the plane is
\[
\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 1 \\ z \end{pmatrix} = 0
\]
or
\[
y - x = 0.
\]
Problem 3. (20 points) Find the first three nonzero terms of the Maclaurin series for the function \( f(x) = \sin(x) \ln(1 + x) \) and find the values of \( x \) for which the series converges absolutely.

Solution: We have:

\[
\sin(x) \log(1 + x) = \left( x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots \right) \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right) \\
= x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \cdots
\]
Problem 4. (20 points) (a) Prove that the limit
\[
\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + y^2}
\]
does not exist.

(b) Show that
\[
\lim_{(x,y) \to (0,0)} \frac{x^2y}{x^2 + y^2} = 0.
\]

Solution: If we approach (0,0) along the path \(y = x\), the function in (a) goes to the value 1. If we approach (0,0) along the path \(y = 0\), the function in (a) goes to the value 0. So no limit at (0,0) can exist.

For (b), we have:
\[
0 \leq \left| \frac{x^2y}{x^2 + y^2} \right| \leq \left| \frac{(x^2 + y^2)y}{x^2 + y^2} \right| = |y|.
\]
Since \(\lim_{(x,y) \to (0,0)} |y| = 0\), it follows that the limit is 0 by the squeezing theorem.
Problem 5. (20 points)

Find the maximum and minimum values of the function \( f(x, y) = e^{xy} \) subject to the constraint \( x^2 + y^2 = 1 \).

Solution: The gradient of \( g(x, y) = x^2 - y^2 - 1 \) is \((2x, 2y)\) and the gradient of \( f(x, y) \) is \((y \exp(xy), x \exp(xy))\). In order for \( f(x, y) \) to have a local minimum or maximum relative to the level surface \( g(x, y) = 0 \) at \((x, y)\) we must have
\[
2x = \lambda y \exp(xy) \\
2y = \lambda x \exp(xy)
\]
for some \( \lambda \neq 0 \). If \( y = 0 \) then these equations imply \( x = 0 \), but the point \((0, 0)\) does not lie on the surface, so we may assume \( y \neq 0 \). We always have \( \exp(xy) > 0 \), so we can rearrange the first equation as
\[
\lambda = \frac{2x}{y \exp(xy)}.
\]
Plugging this into the second equation gives
\[
2y = \frac{2x}{y \exp(xy)} x \exp(xy) = \frac{2x^2}{y}
\]
or \( y^2 = x^2 \). Inserting this into the equation \( x^2 + y^2 = 1 \) for the level surface, we obtain \( x^2 + x^2 = 1 \) or \( x = \pm 1/\sqrt{2} \).

It follows that the maximum and minimum values of \( f(x, y) \) on the surface can occur at one of the points
\[
\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \text{ or } \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).
\]
The minimum is \( \exp(-1/2) \) which occurs at the second and third of these points and the maximum is \( \exp(1/2) \) which occurs at the first and fourth.
**Problem 6.** (20 points) (a) Find the Taylor series for the function \( f(x) = 1/x \) around the point \( x_0 = 1 \).

(b) Find the Taylor series for the function \( g(x) = \log(x) \) around the point \( x_0 = 1 \).

**Solution:** We can find the Taylor series for the function \( f(x) = 1/x \) by computing its derivatives at the point 1. Since:

\[
\begin{align*}
  f'(x) &= -x^{-2} \\
  f''(x) &= 2x^{-3} \\
  f'''(x) &= -6x^{-4} \\
  &\vdots
  \\
  f^{(k)}(x) &= (-1)^k k! x^{-k-1}
\end{align*}
\]

we have \( f'(1) = -1 \) and \( f''(1) = 2 \), so the Taylor series for \( f(x) = 1/x \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n n! (x-1)^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (x-1)^n
\]

\[
= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots
\]
Problem 7. (20 points) Find the absolute maximum and minimum of the function
\[ f(x, y) = x^2 + 4y^2 - 2x^2y + 4 \]
on the rectangle \( R = \{(x, y) : -1 \leq x, y \leq 1\} \).

Solution: We did this problem in class.
Problem 8. (20 points) Find the minimum distance from the surface \(x^2 - y^2 - z^2 = 1\) to the origin.

Solution: Let \(g(x, y, z) = x^2 - y^2 - z^2 - 1\) and \(f(x, y, z) = x^2 + y^2 + z^2\). The problem asks us to minimize \(f(x, y, z)\) on the level surface \(g(x, y, z) = 0\).

If \((x, y, z)\) is a local maximum of \(f(x, y, z)\) on this surface, then we will have \(\nabla f(x, y, z) = \lambda \nabla g(x, y, z)\) for some \(\lambda \neq 0\) and \(g(x, y, z) = 0\); that is,

\[
\begin{align*}
2x &= \lambda 2x \\
2y &= -\lambda 2y \\
2z &= -\lambda 2z \\
x^2 - y^2 - z^2 &= 1.
\end{align*}
\]

We must have \(\lambda = 1\) and \(y = z = 0\). So \(x^2 = 1\) and we see that \((1, 0, 0)\) and \((-1, 0, 0)\) are the possible maximums of the function \(f(x, y, z)\) on the surface. In fact, the maximum value is 1 and it occurs at both of these points.
Problem 9. (20 points) Find all of the local maximums, local minimums and saddle points of the function $f(x, y) = x^3 - y^3 - 2xy + 6$. Make sure you indicate which points are local maximums, which are local minimums and which are saddle points.

Solution: The gradient of $f(x, y)$ is
\[ \nabla f(x, y) = \begin{pmatrix} 3x^2 - 2y \\ -3y^2 - 2x \end{pmatrix}, \]
so the critical points of $f(x, y)$ are $(0, 0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$. The discriminant of $f(x, y)$ is
\[ D(x, y) = -4 - 36xy \]
and
\[ f_{xx}(x, y) = 6x. \]
Since $D(0, 0) < 0$, $(0, 0)$ is a saddle point.

Since $D(-2/3, 2/3) > 0$ and $f_{xx}(-2/3, 2/3) < 0$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$ is a local maximum.
Problem 10. (20 points) Show that if $\{a_n\}$ is a bounded sequence then the power series
\[ \sum_{n=1}^{\infty} a_n x^n \]
has a positive radius of convergence. (Hint: use the root test.)

Solution: There is a $M > 0$ such that $|a_n| \leq M$ for all $n$. So
\[ |a_n x^n| \leq M|x^n| \]
for all $n$. By the root test, the series
\[ \sum_{n=0}^{\infty} M|x|^n \]
converges for any $|x| < 1$ since
\[ \lim_{n \to \infty} (|x|^n M)^{1/n} = |x| \lim_{n \to \infty} M^{1/n} = |x|. \]
So we have by the comparison test that the series $\sum a_n x^n$ converges absolutely for any $|x| < 1$. It follows that the radius of convergence of the series is at least 1.