WRITE YOUR NAME AND SECTION NUMBER ON THE FRONT COVER OF THE EXAM RIGHT NOW.

This is a closed note exam. No books, notes or other references are allowed.

No calculators are allowed.

No cell phones or other electronic devices are allowed.

Please write your answers directly on this exam.

Name: _____________________________ _____________________________

Section: _____________________________ _____________________________
Problem 1. (10 points) Find the limits of the following sequences. Justify your answers.

(a) \[ \lim_{n \to \infty} \frac{\sin(n)}{n} \]

(b) \[ \lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n \]

Solutions: (a) This sequence converges to 0 by the sandwich theorem since
\[ \left| \frac{\sin(n)}{n} \right| \leq \left| \frac{1}{n} \right| \to 0 \text{ as } n \to \infty. \]

(b) We begin by writing:
\[ \lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = \lim_{n \to \infty} \exp \left( \log \left( \left(1 + \frac{2}{n}\right)^n \right) \right) = \lim_{n \to \infty} \exp \left( n \log \left(1 + \frac{2}{n}\right) \right). \]

Since \(\exp(x)\) is continuous we have:
\[ \lim_{n \to \infty} \exp \left( n \log \left(1 + \frac{2}{n}\right) \right) = \exp \left( \lim_{n \to \infty} n \log \left(1 + \frac{2}{n}\right) \right). \]

We now apply L’Hopital’s rule to see that
\[ \lim_{n \to \infty} n \log \left(1 + \frac{2}{n}\right) = \lim_{n \to \infty} \frac{\log \left(1 + \frac{2}{n}\right)}{1/n} = \lim_{n \to \infty} \frac{\log \left(1 + \frac{2}{n}\right)}{1/n} = \lim_{n \to \infty} \frac{2}{1 + \frac{2}{x}} = 2. \]

Putting this together gives:
\[ \lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = e^2. \]
Problem 2. (10 points) Find the values of the following series.

(a) \[ \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \]

(b) \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \]

Solution: (a) Note that the sum is from 1 to \(\infty\), not 0 to \(\infty\). We can compute its value by using the usual formula for geometric series but we must be careful to account for the missing constant term:

\[ \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = -1 + \sum_{n=0}^{\infty} \left( \frac{1}{4} \right)^n = -1 + \frac{1}{1 - \frac{1}{4}} = \frac{1}{3}. \]

(b) This is a telescoping series since

\[ \frac{1}{n^2 + n} = \frac{1}{n} - \frac{1}{n + 1}. \]

We can see that the value of this sum is 1 by observing that the partial sum \(S_n\) is

\[ S_n = 1 - \frac{1}{n + 1} \]

and taking the limit directly.
Problem 3. (20 points) Determine whether each of the following series converge or diverge. Give reasons for your answers.

(a) \[ \sum_{n=0}^{\infty} \frac{1}{n^2 + 2n + 1} \]

(b) \[ \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^n \]

(c) \[ \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)} \]

(d) \[ \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \]

Solution: (a) Since 
\[ \lim_{n \to \infty} \frac{1}{n^2 + 2n + 1} = \lim_{n \to \infty} \frac{n^2}{n^2 + 2n + 1} = \lim_{n \to \infty} \frac{2n}{2n + 2} = \lim_{n \to \infty} \frac{2}{2} = 1 \]
and \( \sum \frac{1}{n^2} \) is convergent, it follows that the series (a) is convergent from the Limit Comparison Test.

(b) This is a geometric series — one of the form \( \sum_{n=0}^{\infty} r^n \) — and it converges since \(|r| = \left| \frac{1}{\sqrt{2}} \right| < 1 \).

(c) This series is convergent by the Alternating Series Convergence Test since \( \ln(x) \) is a positive, monotonically decreasing function which converges to 0.

(d) Since 
\[ \lim_{n \to \infty} \frac{\ln(n)}{n^{3/2}} = \lim_{n \to \infty} \frac{\ln(n)}{n \cdot n^{1/2}} = \lim_{n \to \infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n \to \infty} \frac{1}{\sqrt{n} \ln(n)} = \lim_{n \to \infty} \frac{2}{\sqrt{n}} = 0 \]
and \( \sum \frac{1}{n^{3/2}} \) converges, the Limit Comparison Test shows that the series (d) converges.
**Problem 4.** (20 points) Find the radius of convergence and interval of convergence of the power series

\[ \sum_{n=1}^{\infty} \frac{n^{1/n}}{4^n} x^n \]

Justify your answer.

**Solution:** We begin by applying the root test:

\[
\left( \frac{n^{1/n}}{4^n} |x|^n \right)^{1/n} = \frac{|x|}{4} n^{1/(n^2)} \to \frac{|x|}{4} \text{ as } n \to \infty.
\]

Note that

\[
\lim_{n \to \infty} n^{1/(n^2)} = \lim_{n \to \infty} \exp \left( \frac{\log(n)}{n^2} \right) = \exp \left( \lim_{n \to \infty} \frac{\log(n)}{n^2} \right) = \exp(0) = 1.
\]

It follows that the series converges for \(|x|/4 < 1\), or \(|x| < 4\).

We now know that the radius of convergence of the series is 4. To find the interval of convergence we need to test the endpoints of the interval \((-4,4)\).

When \(x = 4\), the series becomes

\[ \sum_{n=1}^{\infty} \frac{n^{1/n}}{4^n} 4^n = \sum_{n=1}^{\infty} n^{1/n} \]

which diverges by the \(n\)th term test since \(\lim_{n \to \infty} n^{1/n} = 1\).

When \(x = -4\), the series becomes

\[ \sum_{n=1}^{\infty} \frac{n^{1/n}}{4^n} (-4)^n = \sum_{n=1}^{\infty} (-1)^n n^{1/n}. \]

This series also diverges by the \(n\)th term test.

So the interval of convergence is \((-4,4)\).
Problem 5. (20 points) (a) Find the Taylor series for the function \( f(x) = \frac{x^2}{2-x} \) around the point \( x_0 = 1 \).

(b) Find the interval of convergence of that Taylor series. Justify your answer.

Solution: (a) It would be difficult to compute the values of the derivatives of \( f(x) \) at the point \( x_0 = 1 \), so we form the series by manipulation. First, we note that
\[
\frac{1}{2-x} = \frac{1}{1-(x-1)} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \cdots = \sum_{n=0}^{\infty} (x-1)^n.
\]

Now we need a Taylor series for the function \( g(x) = x^2 \) around the point \( x_0 = 1 \). Note that \( x^2 \) is a power series for this function around the point 0 not around the point 1! We can do this either through algebraic manipulation or by computing the derivatives of \( g(x) \) at \( x_0 = 1 \). I prefer the later approach. Since
\[
\begin{align*}
g(1) &= 1 \\
g'(1) &= 2 \\
g''(1) &= 2 \\
g^{(n)}(1) &= 0 \quad \text{for } n > 2,
\end{align*}
\]
we have that
\[ x^2 = g(x) = 1 + 2(x-1) + (x-1)^2. \]
You can verify this identity.

Now we take the product of the two power series:
\[
\frac{x^2}{2-x} = (1 + 2(x-1) + (x-1)^2) \sum_{n=0}^{\infty} (x-1)^n
\]
\[ = \sum_{n=0}^{\infty} (x-1)^n + 2(x-1) \sum_{n=0}^{\infty} (x-1)^n + (x-1)^2 \sum_{n=0}^{\infty} (x-1)^n \]
\[ = \sum_{n=0}^{\infty} (x-1)^n + 2 \sum_{n=1}^{\infty} (x-1)^n + \sum_{n=2}^{\infty} (x-1)^n \]
\[ = 1 + 3(x-1) + \sum_{n=2}^{\infty} 4(x-1)^n. \]

(b) We can find the radius of convergence by applying the root test to the third term in the last line of equation (1) since we can always neglect a finite number of leading terms without affecting convergence. Since
\[
(4|x-1|^n)^{1/n} = 4^{1/n} |x| \rightarrow |x-1| \quad \text{as } n \rightarrow \infty,
\]
the series converges when \( |x-1| < 1 \); that is, it converges on the interval \((0, 2)\).

When \( x = 0 \), the series becomes
\[ 1 + 3(-1) + \sum_{n=2}^{\infty} 4(-1)^n, \]
which diverges by the $n$th term test. When $x = 2$, the series becomes

$$1 + 3(2) + \sum_{n=2}^{\infty} 4(2)^n,$$

which also diverges by the $n$th term test.

So the interval of convergence is $(0, 2)$. 
Problem 6. (20 points) (a) Find the Maclaurin series for the function
\[ f(x) = \frac{1}{1 + x^2}. \]
(b) Find the Maclaurin series for the function
\[ \arctan(x). \]
(c) Find the Maclaurin series for the function
\[ \arctan(x^2). \]

Solution: (a) We use the formula for geometric series:
\[
\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.
\]
(b) Since \( \arctan(x) \) is an antiderivative of \( f(x) \), we can find a Maclaurin series for \( \arctan(x) \) through term-by-term integration:
\[
\arctan(x) = \int \frac{1}{1 + x^2} \, dx
= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx
= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \, dx
= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.
\]
(c) We can substitute \( x^2 \) into the expression we just obtained to find the power series for \( \arctan(x^2) \):
\[
\arctan(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}
= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}.
\]