Problem One (5 points). Find the interval of convergence of the series
\[ \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n. \]

**Solution:** We begin by applying the root test to the sequence \( a_n = (1 + \frac{1}{n})^n |x|^n; \)
\[ (a_n)^{1/n} = \left(1 + \frac{1}{n}\right) |x| \rightarrow |x| \text{ as } n \rightarrow \infty. \]
So the series converges absolutely for \( |x| < 1. \) We now know the radius of convergence, but not the interval of convergence because it might or might not converge at \( x = 1 \) and \( x = -1. \)

When \( x = 1 \) the series becomes
\[ \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n. \]
This diverges by the \( n \)th term test since
\[ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e. \]
This calculation follows from L’Hopital’s rule (for instance) and we worked it out in class.

When \( x = -1, \) we have the alternating series
\[ \sum_{n=1}^{\infty} (-1)^n b_n \quad \text{where} \quad b_n = \left(1 + \frac{1}{n}\right)^n. \]
But \( b_n \rightarrow e \) as \( n \rightarrow \infty \) so the \((-1)^n b_n\) does not converge to 0. So this series also diverges by the \( n \)th term test.

We now know that the interval of convergence of the series is \((-1, 1).\)

Problem Two (5 points). Does the series
\[ \sum_{n=1}^{\infty} a_n, \]
where \( a_n \) is defined recursively by
\[ a_1 = 2, \quad a_{n+1} = \frac{a_n}{n} \quad \text{for } n > 1, \]
converge or diverge?

**Solution:** Since
\[ \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1, \]
it follows that \( \sum_{n=1}^{\infty} a_n \) converges by the ratio test.