**Problem:** Suppose that \( f(x) \) is continuous on \([c, d]\) and \( g(x) \) is a Riemann integrable function on \([a, b]\) such that \( g([a, b]) \subset [c, d] \). Show that \( f(g(x)) \) is Riemann integrable on \([a, b]\).

**Solution:** The function \( f \) is continuous on the compact interval \([c, d]\), so we can choose \( \delta > 0 \) such that for all \( x \) and \( y \) in \([c, d]\)

\[
|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\epsilon}{2(b - a)}.
\]

Moreover, \( f \) is bounded on the range of \( g \), so there exists a \( M \) such that \( |f(x)| < M \).

Choose a partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \([a, b]\) such that

\[
\sum_{j=1}^{n} \omega_j(g)(x_j - x_{j-1}) \leq \frac{\epsilon \delta}{4M},
\]

where \( M_j(h) \) (\( m_j(h) \)) denotes as usual the supremum (infimum) of a function \( h \) over \([x_{j-1}, x_j]\) and \( \omega_j(h) \) is defined by

\[
\omega_j(h) = M_j(h) - m_j(h) = \sup_{x_{j-1} \leq x \leq x_j} (h(x) - h(y))
\]

(this quantity is often referred to as the oscillation of \( h \) on \([x_j, x_{j-1}]\)).

We seek a bound for the sum

\[
\sum_{j=1}^{n} w_j(f \circ g)(x_j - x_{j-1})
\]

We do this by dividing the subintervals \([x_{j-1}, x_j]\) into two parts: those for which \( w_j(g) \geq \delta \) and those for which \( w_j(g) < \delta \):

\[
\sum_{j=1}^{n} w_j(f \circ g)(x_j - x_{j-1}) = \sum_{w_j(g) \geq \delta} w_j(f \circ g)(x_j - x_{j-1}) + \sum_{w_j(g) < \delta} w_j(f \circ g)(x_j - x_{j-1}).
\]

Now

\[
\sum_{w_j(g) < \delta} w_j(f \circ g)(x_j - x_{j-1}) \leq \sum_{w_j(g) < \delta} \frac{\epsilon}{2(b - a)}(x_j - x_{j-1}) \leq \frac{\epsilon}{2}
\]

follows from (1).

Observe that

\[
\sum_{w_j(g) \geq \delta} w_j(g)(x_j - x_{j-1}) \geq \sum_{w_j(g) \geq \delta} \delta(x_j - x_{j-1}) = \delta \sum_{w_j(g) \geq \delta} (x_j - x_{j-1}),
\]

from which it follows that

\[
\sum_{w_j(g) \geq \delta} (x_j - x_{j-1}) \leq \frac{\epsilon}{4M}.
\]

Thus we have

\[
\sum_{w_j(g) \geq \delta} w_j(f \circ g)(x_j - x_{j-1}) \leq \sum_{w_j(g) \geq \delta} 2M \cdot \omega_j(x_j - x_{j-1}) \leq \frac{\epsilon}{2}.
\]
This completes the proof since we now have

\[
\sum_{j=1}^{n} w_j (f \circ g)(x_j - x_{j-1}) = \sum_{w_j(g) \geq \delta} w_j (f \circ g)(x_j - x_{j-1}) + \sum_{w_j(g) < \delta} w_j (f \circ g)(x_j - x_{j-1}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\]