1. Prove that if $f : [0, 1] \rightarrow \mathbb{R}$ is integrable and $f(q) = 0$ for every rational number $q$ in $[0, 1]$, then
\[ \int_0^1 f(x)dx = 0. \]

**Solution:** Since $f$ is Riemann integrable, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$ such that
\[ \left| \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) - \int_0^1 f(x)dx \right| \leq \epsilon \]
for any choice of $t_j$ in $[x_{j-1}, x_j]$. We can choose the $t_j$ to be rational numbers, which leads to the inequality
\[ \left| \int_0^1 f(x)dx \right| \leq \epsilon \]
(1)

since $f(t_j) = 0$ for all $j$. Since the inequality (1) holds for all $\epsilon > 0$, the integral must be 0.
2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Prove that

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt$$

(Hint: differentiate both sides).

**Solution:** Let

$$g(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt.$$  

Clearly $g(0) = f(0)$. We will show that $g'(x) = f'(x)$ for all $x$, which will establish the claim. We write:

$$g(x) = f(0) + f'(0)x + x\int_0^x f''(t)dt - \int_0^x tf''(t)dt$$

and apply the fundamental theorem (and product rule) to obtain

$$g'(x) = f'(0) + \int_0^x f''(t)dt + xf''(x) - xf''(x)dt$$  

$$= f'(0) + \int_0^x f''(t)dt.$$  

We apply the fundamental theorem again, this time to the integral on the right, to obtain

$$g'(x) = f'(0) + f'(x) - f'(0) = f'(x).$$
3. Suppose that function \( g : \mathbb{R}^2 \to \mathbb{R} \) satisfies the inequality

\[ |g(x, y)| \leq x^2 + y^2 \]

for all \((x, y)\) in \( \mathbb{R}^2 \).

(a) What is the value of \( g(0,0) \)?

(b) What are the values of the first order partial derivatives of \( g(x, y) \) at \((0,0)\)?

(c) Show that \( g(x, y) \) is differentiable at \((0,0)\).

**Solution:**

(a) That \( g(0, 0) = 0 \) follows immediately from the inequality.

(b) For any \( h \),

\[
\left| \frac{g(h, 0) - g(0, 0)}{h} \right| \leq \frac{|h|^2}{|h|} = |h| \to 0 \text{ as } h \to 0.
\]

That is,

\[
\frac{\partial f}{\partial x}(0, 0) = 0.
\]

A similar calculation shows that

\[
\frac{\partial f}{\partial y}(0, 0) = 0.
\]

(c) Since the first order partials of \( g(x, y) \) are 0 at \((0,0)\), if \( g \) is differentiable its linear approximate must be the zero transformation. So we need to show that

\[
\lim_{\|h\| \to 0} \frac{|g(h) - g(0,0)|}{\|h\|} = 0.
\]

From the fact that \( g(0,0) = 0 \) and the inequality, we have:

\[
\frac{|g(h) - g(0,0)|}{\|h\|} = \frac{|g(h)|}{\|h\|} \leq \frac{|h|^2}{\|h\|} = \|h\| \to 0 \text{ as } \|h\| \to 0.
\]
4. For which values of $\alpha > 0$ is the function

$$f(x) = \frac{1}{x \log^{\alpha}(x)}$$

improperly integrable on $[e, \infty)$?

**Solution:** Under the substitution $u = \log(x)$, the integral becomes

$$\int_{1}^{\infty} \frac{1}{u^{\alpha}} \, du$$

since $dx = x \, du$. The second integrand is improperly integrable for $\alpha > 1$ as we have shown in class.
5. Prove or disprove: if \( f \) is improperly integrable on \([1, \infty)\) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a bounded function, then the product function \( f(x)g(x) \) is improperly integrable on \([1, \infty)\). (Hint: be careful and think of the examples we discussed in class)

**Solution:** The claim is **not** true. Let

\[
f(x) = \frac{\sin(x)}{x}
\]

and define \( g(x) \) by

\[
g(x) = \begin{cases} 
-1 & \text{if } \sin(x) < 0 \\
0 & \text{if } \sin(x) = 0 \\
1 & \text{if } \sin(x) > 0
\end{cases}
\]

Then \( g(x) \) is a bounded function but for \( x > 0 \)

\[
\left| \frac{g(x)\sin(x)}{x} \right| = \left| \frac{\sin(x)}{x} \right|.
\]

Since \( f(x) \) is improperly integrable on \([1, \infty)\) but \( g(x)f(x) \) is not, the claim cannot be true.

Note: we proved in class that \( f(x) \) is improperly integrable (which can be verified easily by parts) and that \( f(x)g(x) \) is not (which took a little more effort; but the proof is in the book).