10.2.8. Give an example of a power series 
\[ \sum_{k=0}^{\infty} a_k x^k \] with interval of convergence exactly 
\( [-\sqrt{2}, \sqrt{2}] \)

Ans.: 
\[ \sum_{k=0}^{\infty} \frac{x^k}{k \cdot 2^k} \]

10.2.12. If the power series \( \sum_{k=0}^{\infty} a_k x^k \) has a radius of convergence \( R \), what must be the radius of convergence of the series \( \sum_{k=0}^{\infty} a_k x^{2k} \)?

Ans.: \( \sqrt{R} \)

11.7.2. Show that continuity of \( f \) at \( x_0 \) does not depend on which of the norms \( \| \cdot \|_1, \| \cdot \|_l, \| \cdot \|_{\infty} \) is in use.

Pf: First check that the following relation defined on norms of \( \mathbb{R}^n \) is an equivalence relation

\[ \| x \|_a \sim \| x \|_b \iff \exists \alpha, \beta > 0 \text{ s.t. } \beta \| x \|_b \leq \| x \|_a \leq \alpha \| x \|_b \] \hspace{1cm} (\ast) \]

It is easy to check that (\ast) implies that \( \| x \|_a \) and \( \| x \|_b \) defines the same notion of continuity.
So we just need to show that $\| \cdot \| \sim \| \cdot \|_1$ and $\| \cdot \| \sim \| \cdot \|_\infty$.

Well we have for any $x \in \mathbb{R}^n$

$$\frac{1}{\sqrt{n}} \| x \|_1 \leq \| x \| \leq \| x \|_1 \quad \text{and} \quad \| x \|_\infty \leq \| x \| \leq \sqrt{n} \| x \|_1$$

13.2.4.

Def 13.1 $\Rightarrow$ 13.2.4

(a) is just 1.

(b) is just 2.

(c) is implied by 3.4.

13.2.4 $\Rightarrow$ Def 13.1

1 is just (a).

2 is just (b).

3 is implies by the following consideration:

(c) says $d(x,y) \leq d(z,x) + d(z,y)$ $\forall x,y,z \in X$.

In particular $d(x,y) \leq d(y,x)$ thus $d(x,y) \leq d(y,x)$.

A symmetric argument shows that $d(y,x) \leq d(x,y)$ thus $d(x,y) = d(y,x)$.

4 follows from 3 and (c).
13.4.7. If \( \{x_n\}, \{y_n\} \) are convergent sequences in \((X, d)\), show that \( \lim_{n \to \infty} d(x_n, y_n) \) exists.

**Pf:** Suppose \( \lim_{n \to \infty} x_n = \tilde{x} \) and \( \lim_{n \to \infty} y_n = \tilde{y} \). I claim that

\[
\lim_{n \to \infty} d(x_n, y_n) = d(\tilde{x}, \tilde{y})
\]

Well \( d(x_n, y_n) \leq d(x_n, \tilde{x}) + d(\tilde{x}, \tilde{y}) + d(y_n, \tilde{y}) \)

by given condition \( \exists N_1 \ s.t. \ for \ n > N_1 \ d(x_n, \tilde{x}) < \frac{\varepsilon}{3} \)

\( \exists N_2 \ s.t. \ for \ n > N_2 \ d(y_n, \tilde{y}) < \frac{\varepsilon}{3} \)

thus for \( n > \max \{ N_1, N_2 \} \)

\[
d(x_n, y_n) \leq d(\tilde{x}, \tilde{y}) + \varepsilon \quad \square
\]

13.6.1 Let \((X, d)\) be a discrete space

a) What function \( f : X \to \mathbb{R} \) are continuous everywhere.

**ans:** All functions are continuous everywhere

**Hint:** \( \forall \varepsilon > 0 \ \text{take} \ \delta = \frac{1}{2} \).

b) What function \( f : \mathbb{R} \to X \) are continuous everywhere

**ans:** Only constant functions are continuous everywhere

**Pf:** Given \( f \) continuous, we have \( \mathbb{R} = \bigcup_{x \in X} f^{-1}(x) \)

given \( f \) being continuous. We know \( f^{-1}(x) \) is an open set in \( \mathbb{R} \) and \( f^{-1}(x) \cap f^{-1}(y) = \emptyset \) if \( x \neq y \). But this is possible only in the case that \( \exists x \ s.t. f^{-1}(x) = \mathbb{R} \) and \( f^{-1}(y) = \emptyset \ \forall y \neq x \ \square \)