2.8.2. If \( \{S_n\} \) is a sequence all of whose values lies inside an interval \([a, b] \), prove that \( \{S_n/n\} \) is convergent.

\[ \text{Pf: Let } \alpha = \max \{|a|, |b|\}, \text{ then } -\alpha \leq a \leq b \leq \alpha, \text{ the sequence } \{\frac{\alpha}{n}\} \text{ converges to } 0, \text{ and the sequence } \{\frac{-\alpha}{n}\} \text{ converges to } 0. \text{ I also have } -\frac{\alpha}{n} \leq \frac{S_n}{n} \leq \frac{\alpha}{n} \text{ thus } \]

\[ \lim_{n \to \infty} \frac{S_n}{n} = 0. \quad \square \]

29.2. Define a sequence \( \{t_n\} \) recursively by setting \( t_1 = 1 \) and \( t_n = \sqrt{t_{n-1} + 1} \). Does this sequence converge? To what?

\[ \text{Solution: Comparing } t_n \text{ and } t_{n+1} = \sqrt{t_n + 1}. \text{ We see that for } t_n \in ( \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} ), \ t_{n+1} > t_n. \text{ Given that } t_1 = 1 \text{ we know the sequence } \{t_n\} \text{ is increasing.} \]

\( \{t_n\} \) is also bounded, say by 3, this can be shown by induction: \( t_n < 3 \) implies \( t_{n+1} < 4, \text{ thus } t_n = \sqrt{t_n + 1} < 2 < 3 \)

Thus we know that \( \{t_n\} \) converges.

\[ \text{Suppose that } t_n \to A, \text{ then we have } \]

\[ A = \sqrt{A + 1} \]

\[ \text{thus } A = \frac{1 + \sqrt{5}}{2}. \text{ We know } A > 1, \text{ thus } A = \frac{1 + \sqrt{5}}{2}. \quad \square \]
2.11.3. If \( \{S_{n_k}\} \) is a subsequence of \( \{S_n\} \), \( \{t_{n_k}\} \) is a subsequence of \( \{t_n\} \), then is it true that \( \{S_{n_k}+t_{n_k}\} \) is a subsequence of \( \{S_n+t_n\} \)?

Solution: No. Say \( \{S_n\} = \{0, 1, 0, 1, 0, 1, \ldots\} \) and \( \{S_{n_k}\} = \{0, 0, 0, \ldots\} \) and \( \{t_n\} = \{1, 0, 1, 0, 1, \ldots\} \) and \( \{t_{n_k}\} = \{0, 0, 0, \ldots\} \).

But \( \{S_{n_k}+t_{n_k}\} = \{1, 1, 1, \ldots\} \)

\( \{S_{n_k}+t_{n_k}\} = \{0, 0, 0, \ldots\} \).

2.11.6.

a) A sequence is convergent iff all of its subsequence are convergent. True

b) A sequence is bounded iff all of its subsequence are bounded. True

c) A sequence is monotone iff all of its subsequence are monotone.

d) A sequence is divergent iff all of its subsequence are divergent.
5.12.3. Show that the sum of two Cauchy sequence is a Cauchy sequence. 

**Proof:** Suppose \( \{a_n\}, \{b_n\} \) are Cauchy, i.e., \( \exists N_a(\varepsilon), N_b(\varepsilon) \) for any \( n, m > N_a(\varepsilon) \) \( |a_n - a_m| < \varepsilon \) \( \forall \varepsilon > 0 \)

for any \( n, m > N_b(\varepsilon) \) \( |b_n - b_m| < \varepsilon \) \( \forall \varepsilon > 0 \).

Now pick \( \varepsilon > 0 \), take \( N = \max \{N_a(\frac{\varepsilon}{2}), N_b(\frac{\varepsilon}{2})\} \) for any \( n, m > N \).

\[
|a_n + b_n - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)| \\
\leq |a_n - a_m| + |b_n - b_m| \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

\( \blacksquare \)

5.16. Show that \( \lim_{x \to 0} \frac{1}{x} \) does not exist by using the sequential definition of limit.

**Solution:** Consider the following two sequence \( \{\frac{1}{n}\} \) and \( \{-\frac{1}{n}\} \).

Clearly \( \frac{1}{n} \to 0 \) and \( -\frac{1}{n} \to 0 \) but

\[
\lim_{n \to \infty} \frac{|1|}{\frac{1}{n}} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{|-\frac{1}{n}|}{\frac{1}{n}} = -1.
\]

\( \blacksquare \)
\[ \lim_{k \to \infty} X_k = L_1 \quad \lim_{k \to \infty} Y_k = L_2. \]

Show that if \( X_n \leq Y_n \) then \( L_1 \leq L_2 \).

**Pf.** Suppose not, \( L_1 > L_2 \). Then let \( \varepsilon = L_1 - L_2 > 0 \).

\( \exists N_x \) s.t. \( \forall n > N_x \ |X_n - L_1| < \frac{\varepsilon}{4} \)

\( \exists N_y \) s.t. \( \forall n > N_y \ |Y_n - L_2| < \frac{\varepsilon}{4} \)

then \( \forall n > \max \{N_x, N_y\} \)

then \( X_n - Y_n = (X_n - L_1 + L_1) - (Y_n - L_2 + L_2) \)

\[ = (X_n - L_1) - (Y_n - L_2) + L_1 - L_2 \]

\[ \geq -\frac{\varepsilon}{4} - \frac{\varepsilon}{4} + \varepsilon = \frac{\varepsilon}{2} > 0 \quad \Rightarrow \]

Note that if \( X_n < Y_n \) it is possible that \( L_1 = L_2 \)

Consider \( X_n = \frac{1}{5^n} \quad Y_n = \frac{1}{n} \)

Clearly \( X_n < Y_n \) but

\( \lim_{n \to \infty} X_n = \lim_{n \to \infty} Y_n = 0. \)

\[ \square \]