5.4.8. Prove that the function \( f(x) = |x| \) is continuous at every point of \( \mathbb{R} \) using the \( \delta-\varepsilon \) form of continuity.

**Pf:** Pick \( \delta > 0 \) and let us consider

\[
|f(x+\delta) - f(x)| = |x+\delta| - |x| \\
= |x+\delta| - |1-x| \\
\leq |x+\delta - x| \quad \text{triangular inequality} \\
\leq (|x|+|\delta|) \leq |x|+|\delta| \\
= \delta
\]

This says that the distance between \( f(x_0) \) and \( f(x) \) is always bounded by the distance of \( x_0 \) and \( x \), that is

\[ |f(x) - f(x_0)| \leq |x_0 - x| \]

Formally, \( \forall x_0 \in \mathbb{R}, \forall \varepsilon > 0 \), let \( \delta = \varepsilon \) then \( |x_0 - x| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \)

\( \square \)

5.5.1. \( f, g \) are functions and \( f+g \) is continuous. Does it imply that at least one of \( f \) and \( g \) is continuous?

**Solution:** No! Consider

\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational}
\end{cases}
\]

\[g(x) = -f(x)\]

Then \( f, g \) are not continuous while \( -f(x) + g(x) = 0 \) is continuous.

\( \square \)
5.5.2. $|f(x)|$ is continuous. Does it follow that $f$ is continuous?

Solution: No. Consider

$$f(x) = \begin{cases} 
1 & x \text{ is rational} \\
-1 & x \text{ is irrational}.
\end{cases}$$

5.6.3. Give an example of a continuous function on $(0, 1)$ that is bounded, but not uniformly continuous.

Solution: Consider $f(x) = \sin \left( \frac{1}{x} \right)$ for $x \in (0, 1)$. This function is clearly bounded, $|f(x)| \leq 1$. So I just need to show that $f$ is not uniformly continuous.

Well, let $\varepsilon = \frac{1}{3}$ for any $\delta > 0$. Let $x = \min \{ \delta, \frac{1}{2} \}$
then there exist $y \in \left( \frac{1}{x + 2\delta}, x \right)$ s.t. $|\sin \left( \frac{1}{y} \right) - \sin \left( \frac{1}{x} \right)| > \frac{1}{3}$

5.6.9. Show that Lipschitz $\Rightarrow$ uniform continuity but not vice versa.

Proof (PF): Given that $f$ is Lipschitz, pick $\delta = \frac{1}{M} \varepsilon$, $|x - x_0| < \frac{1}{M} \varepsilon$

then $|f(x) - f(x_0)| < M |x - x_0| < \varepsilon$

There exists functions that are uniformly continuous but not Lipschitz. For example $y = \sqrt{x}$, $x \in [0, 1]$
Show that the image of a compact set under a continuous function is compact.

pf: Let \( C \) be a compact set and \( f \) be a continuous function on \( C \). We want to show that \( f(C) \) is compact.

Let \( \{a_n\} \) be an arbitrary sequence in \( f(C) \). Consider the sequence \( \{b_n\} \) of \( C \) defined by

\[
b_k = \inf f^{-1}(a_k) = \min f^{-1}(a_k)
\]

(f is continuous: thus \( f^{-1}(a_k) \) is closed)

if we don't assume \( C \subseteq \mathbb{R} \) then we need axiom of choice here.

Given that \( C \) is compact, there is a convergent subsequence \( \{b_{n_j}\} \) of \( \{b_n\} \). Then \( \{f(b_{n_j})\} \) is a convergent subsequence of \( \{a_n\} \). (Again, we use the fact that \( f \) is continuous here.)

\( \square \)