1. Please write your name on the cover sheet.
2. Write your answers on the exam paper.
3. You may use one sheet of prepared notes.
4. Calculators, cell phones, and other electronic devices are not allowed.
5. Good luck!
1. Find all local maxima and minima of the function \( f(x) = x^2e^{-x} \) and indicate on which intervals \( f(x) \) is increasing and on which intervals it is decreasing. Be sure to indicate which critical points are local maxima and which are local minima.

\[
\text{Find } f'(x): \quad f'(x) = x^2(-e^{-x}) + 2xe^{-x} = (2x-x^2)e^{-x}
\]

\[
\text{Find critical points: } f'(x) = 0 \Rightarrow (2x-x^2)e^{-x} \Rightarrow x(2-x) = 0 \Rightarrow x = 0 \text{ or } 2
\]

\[
\text{Find increasing/decreasing and classify critical points:}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (x) )</th>
<th>( (2-x) )</th>
<th>( e^{-x} )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty, 0))</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>(0, 2))</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(2, \infty))</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

\[
\text{f(x) is increasing on } (0, 2), \text{ decreasing on } (-\infty, 0) \cup (2, \infty)
\]

\[
(0, 0) \text{ is a local minimum, } (2, 4e^{-2}) \text{ is a local maximum.}
\]
2. Find the following antiderivatives:

a. \( \int \left( \frac{1}{\sqrt{x}} + \sqrt{x} \right) \, dx \)

b. \( \int \frac{(1+x^3)^2}{x} \, dx \)

c. \( \int \frac{2x}{x^2+1} \, dx \)

\[ \text{a} = \int \frac{dx}{\sqrt{x}} + \int \sqrt{x} \, dx = \int x^{-\frac{1}{2}} \, dx + \int x^{\frac{1}{2}} \, dx = 2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C \]

\[ \text{b} = \int \frac{1 + 2x^3 + x^6}{x} \, dx = \int \frac{dx}{x} + \int 2x^2 \, dx + \int x^5 \, dx = \ln |x| + \frac{2}{3}x^3 + \frac{x^6}{6} + C \]

\[ \text{c} = \int \frac{2x \, dx}{x^2+1} = \int \frac{du}{u} = \ln |u| + C = \ln |x^2+1| + C = \ln(x^2+1) + C \] (since \( x^2+1 > 0 \) for any \( x \))
3. Suppose that \( \frac{d}{dx} F(x) = f(x) \).

Write

\[ \int f(x^3) x^2 \, dx \]

in terms of \( F(x) \).

\[ \text{Since } \frac{d}{dx} F(x) = f(x), \quad \int f(x) \, dx = F(x) + C \]

(\( \text{since } F(x) \text{ is an antiderivative of } f(x) \)).

\[ \text{Then: } \]

\[ \int f(x^3) x^2 \, dx = \frac{1}{3} \int f(u) \, du = \frac{1}{3} F(u) + C = \frac{1}{3} F(x^3) + C \]

(\( \text{let } u = x^3 \))

\[ du = 3x^2 \, dx \]

\[ \frac{du}{3} = x^2 \, dx \]

(By (*) above)

(You can see this in action by taking an example \( F(x) \):

If we let \( F(x) = x^2 \), then \( f(x) = \frac{d}{dx} F(x) = 2x \).

Then

\[ \int f(x^3) x^2 \, dx = \int 2x^3 \cdot x^2 \, dx = 2 \int x^6 \, dx = \frac{2x^7}{7} + C \]

\[ = \frac{1}{3} (x^3)^2 + C \]

\[ = \frac{1}{3} F(x^3) + C. \]

But...

To get full points, you need the general form of the answer above (in terms of a general \( F(x) \)).)
4. Find the values of the following definite integrals:

a. \[ \int_{\pi/2}^{\pi} \cos(x) \, dx \]

b. \[ \int_{0}^{\sqrt{\pi}} x \sin(x^2) \, dx \]

c. \[ \int_{0}^{\pi^2} \frac{\sin(\sqrt{x})}{2\sqrt{x}} \, dx \]

\[ \text{a)} \quad \int_{\pi/2}^{\pi} \cos x \, dx = \sin x \bigg|_{\pi/2}^{\pi} = 0 - 1 = -1 \]

\[ \text{b)} \quad \int_{0}^{\sqrt{\pi}} x \sin(x^2) \, dx = \frac{1}{2} \int_{0}^{\sqrt{\pi}} \sin(x^2) \cdot 2x \, dx \]
\[ = \frac{1}{2} \int_{0}^{\sqrt{\pi}} \sin(x^2) \, d(x^2) \]
\[ = -\frac{1}{2} \cos(x^2) \bigg|_{0}^{\sqrt{\pi}} = 1 \]

\[ \text{c)} \quad \int_{0}^{\pi^2} \frac{\sin(\sqrt{x})}{2\sqrt{x}} \, dx \]
\[ = \int_{0}^{\sqrt{\pi}} \sin(\sqrt{x}) \, d(\sqrt{x}) \]
\[ = -\cos(\sqrt{x}) \bigg|_{0}^{\pi^2} = -(-1) - (-1) = 2 \]
5. What is the derivative of \( f(x) = x^{\ln(x)} \) when \( x > 0 \). Write your answer in terms of \( x^{\ln(x)} \).

\[
\frac{f'(x)}{f(x)} = \frac{2 \ln x}{x}
\]

\[
\Rightarrow f'(x) = \frac{2 \ln x}{x} \cdot x \ln x
\]
6. Let

\[ g(x) = \int_0^x f'(t) \, dt \]

where \( f(t) = \arctan(t) \). What is \( g(1) \)?

Note: \( \arctan \) is the inverse of the restriction of the tangent function to the interval \([-\pi/2, \pi/2]\).

\[ g(1) = \int_0^1 f'(t) \, dt = f(t) \bigg|_0^1 = f(1) - f(0) \]

because

\[ \tan(0) = 0 \quad \text{and} \quad \tan\left(\frac{\pi}{4}\right) = 1 \]

\[ \implies \quad \arctan(0) = 0 \quad \text{and} \quad \arctan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \]

\[ \implies \quad f(1) - f(0) \]

\[ = \arctan(1) - \arctan(0) \]

\[ = \frac{\pi}{4} \]
7. Find the equation of the line tangent to the curve defined by the equation 

\[ y \ln(y) + x = 0 \]

at the point \((-e, e)\).

\[ \text{Find } y': \]

\[ y' \left( \frac{1}{y} \right) y' + (\ln y) y' + 1 = 0 \]

\[ y' (1 + \ln y) = -1 \]

\[ y' = \frac{-1}{1 + \ln y} \]

\[ \text{Slope of tangent at } (-e, e): \]

\[ y'_{0} = \frac{-1}{1 + \ln e} = \frac{-1}{2} \]

\[ \text{Tangent line at } (-e, e): \]

\[ y - e = \frac{-1}{2} (x + e) \]

\[ \boxed{y = \frac{-1}{2} x + \frac{1}{2} e} \]

\[ \boxed{\text{Ans}} \]