1. (20 points) Find the following indefinite integrals (antiderivatives):

(a) \( \int x\sqrt{x^2 - 1} \, dx \)

(b) \( \int x\sqrt{x - 1} \, dx \)

(c) \( \int x \cos(x - 1) \, dx \)

(d) \( \int \frac{\cos(x)}{1 + e^{\sin(x)}} \, dx \)

The following formulas may be helpful:

\[
\int \frac{1}{\sqrt{u^2 - a^2}} \, du = \ln \left| u + \sqrt{u^2 - a^2} \right| + C
\]

\[
\int \frac{1}{1 + e^u} \, du = u - \ln |1 + e^u| + C
\]

\[
\int \frac{1}{1 + e^{nu}} \, du = u - \frac{1}{n} \ln |1 + e^{nu}| + C
\]

2. (10 points) What is the area of the region bounded by the graphs of \( y = x^3 \) and \( y = 2x - x^2 \)?

3. (10 points) Complete the square in order to use the formula

\[
\int \frac{1}{\sqrt{u^2 - a^2}} \, du = \ln \left| u + \sqrt{u^2 - a^2} \right| + C
\]

to find the antiderivative of

\[
\int (x^2 + 8x - 9)^{-1/2} \, dx
\]

4. (10 points) What is the volume of the solid of revolution formed by rotating the graph of the function

\[
f(x) = \sqrt{\frac{x - 1}{x^2 - 2x - 3}}, \quad 0 \leq x \leq 1
\]

around the x-axis.

5. (10 points) Let \( \beta \) be a positive constant. Write the antiderivative

\[
\int \frac{1}{x^2 - \beta^2} \, dx
\]

in terms of \( \beta \).
1a. Using the substitution \( u = x^2 - 1 \) we rewrite the integral as
\[
\int x\sqrt{x^2 - 1} \, dx = \frac{1}{2} \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (x^2 - 1)^{3/2} + C
\]

1b. We can proceed either with parts or a substitution. I’ll solve the problem using the substitution \( u = x - 1 \), in which case we have:
\[
\int x\sqrt{x - 1} \, dx = \int (u + 1)\sqrt{u} \, du = \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C = \frac{2}{5} (x - 1)^{5/2} + \frac{2}{3} (x - 1)^{3/2} + C
\]

1c. We will use integration by parts here. Let \( u'(x) = \cos(x - 1) \) and \( v(x) = x \). Then \( u(x) = \sin(x - 1) \) and \( v'(x) = 1 \) so by the parts formula we have:
\[
\int x \cos(x - 1) \, dx = \int v(x)u'(x) \, dx = u(x)v(x) - \int v'(x)u(x) \, dx = x \sin(x - 1) - \int \sin(x - 1) \, dx = x \sin(x - 1) + \cos(x - 1) + C
\]

1d. The substitution \( u = \sin(x) \) enables us to use the second formula to find the antiderivative, which is:
\[
\sin(u) - \ln\left|1 + e^{\sin(u)}\right| + C
\]

2. First we need to find the points of intersection of the two graphs, which occur when \( x^3 - 2x + x^2 = 0 \). The roots of that polynomial (which are the points of intersection) are \( x = -2, 0, 1 \). So we need to integrate over the intervals \([-2, 0]\) and \([0, 1]\). One \([-2, 0]\], we have \( x^3 > 2x - x^2 \) and on \([0, 1]\) we have \( x^3 < 2x - x^2 \), so the area is
\[
\int_{-2}^{0} (x^3 - (2x - x^2)) \, dx + \int_{0}^{1}((2x - x^2) - x^3) \, dx
\]
\[
= \int_{-2}^{0} x^3 + x^2 - 2x \, dx - \int_{0}^{1} x^3 + x^2 - 2x \, dx
\]
\[
= \left. \left( \frac{x^4}{4} + \frac{x^3}{3} - x^2 \right) \right|_{-2}^{0} - \left. \left( \frac{x^4}{4} + \frac{x^3}{3} - x^2 \right) \right|_{0}^{1}
\]
\[
= \frac{8}{3} + \frac{5}{12} = \frac{37}{12}
\]
3. We can write \( x^2 + 8x - 9 \) as \( (x + 4)^2 - 25 \) so we have

\[
\int (x^2 + 8x - 9)^{-1/2} \, dx = \int \frac{1}{\sqrt{x^2 + 8x - 9}} \, dx \\
= \int \frac{1}{\sqrt{(x + 4)^2 - 25}} \, dx.
\]

If we let \( u = x + 4 \) and \( a = 5 \) then we see that

\[
\int \frac{1}{\sqrt{(x + 4)^2 - 25}} \, dx = \int \frac{1}{\sqrt{u^2 - a^2}} \, du \\
= \ln |u + \sqrt{u^2 - a^2}| + C \\
= \ln |(x + 4) + \sqrt{(x + 4)^2 - 25}| + C.
\]

4. This is really a partial fractions problem since the volume of the surface of revolution is

\[
\int_0^1 \pi [f(x)]^2 \, dx = \pi \int_0^1 \frac{x - 1}{x^2 - 2x - 3} \, dx.
\]

The denominator of the integrand factors as \((x - 3)(x + 1)\) and we seek \(A\) and \(B\) such that

\[
\frac{x - 1}{(x - 3)(x + 1)} = \frac{A}{x + 1} + \frac{B}{x - 3}.
\]

To find \(A\) and \(B\) we solve the system of equations

\[
A + B = 1 \quad B - 3A = -1,
\]

the unique solution of which is

\[
A = \frac{1}{2}, \quad B = \frac{1}{2}.
\]

It follows that

\[
\int \frac{x - 1}{(x - 3)(x + 1)} \, dx = \int \frac{1}{2(x + 1)} \, dx + \int \frac{1}{2(x - 3)} \, dx \\
= 1/2 \ln |x + 1| + 1/2 \ln |x - 3| + C \\
= 1/2 \ln |x^2 - 2x - 3| + C
\]

and the final answer to the problem is

\[
\int_0^1 \pi [f(x)]^2 \, dx = \frac{\pi}{2} (\ln(4) - \ln(3)) = \frac{\pi}{2} \ln \left(\frac{4}{3}\right).
\]

5. The purpose of this problem is to test if you really understand partial fractions. Its very similar to one of the substitution problems asked on the previous midterm (and there’s one more method of integration I can test you on ... hint hint).

The denominator factors as \((x - \beta)(x + \beta)\) so we wish to find \(A\) and \(B\) such that

\[
\frac{1}{(x - \beta)(x + \beta)} = \frac{A}{x - \beta} + \frac{B}{x + \beta}.
\]
Equation (1) leads to the system of equations
\[ A + B = 0 \quad \beta \cdot A - \beta \cdot B = 1 \]
which has the unique solution
\[ A = \frac{1}{2\beta} \quad B = -\frac{1}{2\beta}. \]
So
\[
\int \frac{1}{x^2 - \beta} \, dx = \frac{1}{2\beta} \left( \int \frac{1}{x - \beta} \, dx - \int \frac{1}{x + \beta} \, dx \right)
\]
\[ = \frac{1}{2\beta} \left( \ln |x - \beta| - \ln |x + \beta| \right) + C \]
\[ = \frac{1}{2\beta} \ln \left| \frac{x - \beta}{x + \beta} \right| + C. \]