5.2.6
a. Suppose that \( g_n \geq 0 \) is a sequence of integrable functions which satisfies
\[
\lim_{n \to \infty} \int_a^b g_n(x) \, dx = 0.
\]
Show that if \( f : [a \to b] \to \mathbb{R} \) is integrable on \([a,b]\), then
\[
\lim_{n \to \infty} \int_a^b f(x) g_n(x) \, dx = 0.
\]
Solution: Since \( f \) is integrable on \([a,b]\), it is bounded on \([a,b]\). Choose \( M \) such that \(|f(x)| < M\) for all \( x \in [a,b] \). Then
\[
\left| \int_a^b f(x) g_n(x) \, dx \right| \leq \int_a^b |f(x)| g_n(x) \, dx \leq \int_a^b M g_n(x) \, dx = M \int_a^b g_n(x) \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]

5.3.1 If \( f : \mathbb{R} \to \mathbb{R} \), find \( F'(x) \) for each of the following functions.

\[ F(x) = \int_0^x f(t-x) \, dt \]

Solution: Applying the change of variables \( u = t - x \), we have
\[
F(x) = \int_{-x}^0 f(u) \, du = -\int_0^{-x} f(u) \, du \quad F'(x) = f(-x).
\]

Problem: Show that if \( g(x) \) is integrable on \([a,b]\) and if \( f(x) \) is a continuous function (continuous on the whole real line) then \( f(g(x)) \) is integrable on \([a,b]\).

Solution: Suppose that \( \epsilon > 0 \). Let \( P_n \) be the partition of \( b - a \) into \( n \) intervals of equal length. We claim that \( U(f \circ g, P_n) - L(f \circ g, P_n) < \epsilon \) for a sufficiently large choice of \( n \).

Since \( g \) is integrable on \([a,b]\), it is bounded on \([a,b]\). Choose \( c,d \) such that \( g(x) \in [c,d] \) for all \( x \in [a,b] \). Because \( f \) is continuous, it is uniformly continuous on \([c,d]\), and we can choose \( \delta > 0 \) such that \( y,z \in [c,d] \) and \( |y - z| < \delta \) imply \(|f(y) - f(z)| < \frac{\epsilon}{2(b-a)}\). For each \( j \) such that \( M_j(g) - m_j(g) < \delta \), it follows that \( M_j(f \circ g) - m_j(f \circ g) < \frac{\epsilon}{2(b-a)} \).
Again using the integrability of $g$, we can choose $n$ such that $U(g, P_n) - L(g, P_n) < \frac{\epsilon \delta}{2(d-c)}$. In this case,

\[
\frac{\epsilon \delta}{2(d-c)} > U(g, P_n) - L(g, P_n) = b - a \sum_{j=1}^{n} (M_j(g) - m_j(g))
\]

\[
\frac{\epsilon \delta n}{2(b-a)(d-c)} > \sum_{j=1}^{n} (M_j(g) - m_j(g)).
\]

Since each term on the right-hand side is nonnegative, there are at most $\frac{\epsilon n}{2(b-a)(d-c)}$ values of $j$ for which $M_j(g) - m_j(g) \geq \delta$. For each of those values of $j$, we must have $M_j(f \circ g) - m_j(f \circ g) \leq d - c$. Furthermore, there are at most $n$ values of $j$ for which $M_j(g) - m_j(g) < \delta$. For each of those values of $j$, we have $M_j(f \circ g) - m_j(f \circ g) < \frac{\epsilon}{2(b-a)}$. We conclude that

\[
U(f \circ g, P_n) - L(f \circ g, P_n) = \frac{b - a}{n} \sum_{j=1}^{n} (M_j(f \circ g) - m_j(f \circ g)) < \frac{b - a}{n} \left[ \frac{\epsilon n}{2(b-a)(d-c)}(d - c) + n \frac{\epsilon}{2(b-a)} \right] = \epsilon.
\]

This completes the proof.