UNIVERSAL QUADRATURES FOR BOUNDARY INTEGRAL EQUATIONS ON TWO-DIMENSIONAL DOMAINS WITH CORNERS

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ABSTRACT. We describe the construction of a collection of quadrature formulae suitable for the efficient discretization of certain boundary integral equations on a very general class of two-dimensional domains with corner points. The resulting quadrature rules allow for the rapid high-accuracy solution of Dirichlet boundary value problems for Laplace’s equation and the Helmholtz equation on such domains under a mild assumption on the boundary data. Our approach can be adapted to other boundary value problems and certain aspects of our scheme generalize to the case of surfaces with singularities in three dimensions. The performance of the quadrature rules is illustrated with several numerical examples.

1. INTRODUCTION

Second kind integral equations arise in the solution of boundary value problems for elliptic partial differential equations with constant coefficients. For instance, a double layer representation

\[ f(x) = \frac{1}{2\pi} \int_{\partial\Omega} \nu(y) \cdot \nabla y \log \|x - y\| \sigma(y) dS(y), \]

where \(\partial\Omega\) is the boundary of a simply-connected planar domain \(\Omega\) and \(\nu(y)\) is the outward unit normal to \(\partial\Omega\) at the point \(y\), for the unique solution to the Dirichlet problem

\[ \Delta f(x) = 0 \text{ for } x \in \Omega \]
\[ \lim_{x \to p} f(x) = u(p) \text{ for } p \in \partial\Omega \]

can be obtained by solving the boundary integral equation

\[ \frac{1}{2}\sigma(x) + \frac{1}{2\pi} \int_{\partial\Omega} \nu(y) \cdot \nabla y \log \|x - y\| \sigma(y) dS(y) = u(x). \] (1.1)

When the boundary \(\partial\Omega\) is smooth, the integral operator

\[ \sigma(x) \to \frac{1}{2\pi} \int_{\partial\Omega} \nu(y) \cdot \nabla y \log \|x - y\| \sigma(y) dS(y) \] (1.2)

is compact and

\[ T\sigma(x) = \frac{1}{2}\sigma(x) + \frac{1}{2\pi} \int_{\partial\Omega} \nu(y) \cdot \nabla y \log \|x - y\| \sigma(y) dS(y) \] (1.3)

is a Fredholm operator, whereas when \(\partial\Omega\) is merely Lipschitz, the operator (1.2) is no longer compact but rather a generalized Calderón-Zygmund operator. The operator \(T\), however, is invertible \(L^2(\partial\Omega) \to L^2(\partial\Omega)\) in either case (see, for instance, [12, 8]).

Clearly, operators of the type (1.3) cannot be represented accurately via any finite rank operator. Discretization, then, can only be effected by explicitly restricting the operator to a finite-dimensional...
subspace. In many applications, the boundary data is known to be the restriction of a smooth function. For instance, it can often be assumed that boundary data is of the form

\[ u(y) = \sum_{j=1}^{N} G(x_j, y) \]

where \( G(x, y) \) is a Green’s function for the partial differential equation and the boundary is distant from the points \( x_j \). This assumption is operative in many acoustic scattering problems (see [9] for a detailed discussion of the solution of acoustic scattering problems via boundary integral methods). In the case of smooth boundaries and smooth boundary data, the discretization of the integral equation (1.1) can be achieved by observing that the solutions are necessarily smooth (see, for instance, [12]). A discrete system of equations approximating (1.1) can then be constructed by representing the desired solutions as polynomials (global or local) and using the appropriate quadratures to approximate the integral portion of the operator (1.3). This approach and its variants are generally known as collocation or Nyström methods (see [2]).

In the case of domains \( \Omega \) whose boundaries \( \partial \Omega \) are Lipschitz, however, solutions of (1.1) are no longer smooth and cannot be represented efficiently using standard techniques. Moreover the evaluation of integrals involving the kernels, which are also singular near corner regions, is much more difficult than in the smooth case. Remedies include the use of improved representations for solutions of boundary integral equations and various classical quadrature schemes for evaluating singular integrals. In [7], for example, asymptotic estimates for the solutions of boundary integral equations arising from Neumann problems for Laplace’s equation are exploited in order to resolve the delicate “cancellation of infinities” which occurs in this case and to develop quadratures for the evaluation of the necessary integrals. Much earlier work (e.g., [2, 3, 15]) focuses on graded meshes and various classical quadrature schemes like subtraction of singularities and substitution.

The scheme of this paper, in contrast, utilizes a very direct mechanism for the efficient discretization of boundary integral equations of the form

\[ \pm \frac{1}{2} I + \int_{\partial \Omega} K(x, y) f(y) dS(y) = u(x) \]

where \( K(x, y) \) is the kernel arising from a Dirichlet boundary value problem for Laplace’s equation or the Helmholtz equation, \( \partial \Omega \) is the boundary of a simply-connected planar domain \( \Omega \) with corners in \( \mathbb{R}^2 \), and the right-hand side \( u(x) \) has the property that it is the restriction to the boundary \( \partial \Omega \) of a function which satisfies the underlying partial differential equation (Laplace or Helmholtz) in a neighborhood of each point on the boundary. This assumption on the right-hand side is equivalent to requiring that \( u(x) \) admits representation as a rapidly converging partial wave expansion (in the case of the Helmholtz equation) or multipole expansion (in the case of Laplace’s equation) and it applies to a number of applications of interest, most notably to certain acoustic scattering problems. Our approach is an extension of that of [6], in which a procedure for the construction of an orthonormal basis spanning the space of restrictions of functions satisfying a Laplace boundary integral equation over a contour \( \Gamma \) to a small curve segment \( \Gamma_0 \subset \Gamma \) is introduced. Using the machinery of [5], such a basis can be exploited to form a “purpose-made” quadrature formula for the efficient discretization of the boundary integral equation over the curve segment \( \Gamma_0 \). As the size of the resulting quadrature rules depends only weakly on the geometry of the curve segment \( \Gamma_0 \), boundary integral equations over curves with complicated geometry (e.g., corners) can be discretized quite efficiently. Moreover, by repeating the basis construction procedure for multiple curve segments, a collection of quadratures suitable for the efficient discretization of a particular boundary integral equation over an entire class of contours with complicated geometry can be precomputed. We refer to such a collection as a set of “universal quadratures.” The construction of universal quadratures for boundary integral equations arising from certain Laplace boundary value problems on polygonal domains is described in [6]. Here, we extend these results to a much more general class of domains with corners and to Helmholtz boundary value problems.

This direct approach has a number of important advantages. First, no \textit{a priori} analytic information about the singularity of solutions is required. This obviates the need for asymptotic estimates of the
behavior of solutions, which are often difficult to derive and not available in many cases of interest. Second, the treatment of various cases is uniform in that no adaptation is required to different corner regions (as is often the case with classical quadrature techniques). Moreover, the precomputation of quadrature formulas eliminates the complexities arising from corner regions almost entirely; that is, standard solvers for the smooth case can be used with minor modification and the number of nodes required to discretize an integral equations over a corner region is reduced to roughly that required to discretize smooth curve regions. Finally, at least in the case of Dirichlet boundary value problems, the resulting discrete systems of equations are well-conditioned. This is a natural consequence of the fact the underlying boundary integral equations are well-behaved (in that their $L^2(\partial\Omega)$ operator norms and those of their inverses are typically small) and their discretizations via quadrature formulas which integrate the correct subspaces inherit this property.

While some of the underlying mechanisms of this paper, particularly the construction of orthonormal bases spanning the space of solutions of integral equations near a corner point, appear to be quite generally applicable, our current implementation suffers from a number of limitations. We assume, for instance, that the boundary data can be represented efficiently as a multipole or partial wave expansion. This excludes many cases of interest involving singular boundary data (see for instance [13] for a discussion of several problems of this type). The constructions of this paper, however, can be easily modified for problems involving more complicated right-hand sides; all that is generally required is an efficient basis spanning the desired space of right-hand sides. More seriously, we encountered difficulties in the solution of Neumann problems for Laplace’s equation. In particular, we encountered ill-conditioned linear systems of equations in this case (see Tables 3, 4 and 5 in Section 7). We are currently investigating this issue. We note that the algorithm of this paper provides a means for accelerating the solution of a boundary value problem assuming the problem can be solved via brute-force and the combination of our algorithm with other techniques for the solution of Laplace Neumann problems should yield a fast accurate algorithm in this case.

The work described in this paper is similar to the scheme proposed by Helsing and Ojala in [13], which appears to be the first paper describing the solution of large-scale problems on domains with corners. In [13], dense meshes of discretization nodes are coupled with a compression scheme for the resulting linear systems in order to rapidly solve boundary integral equations on large-scale domains with corners. The principal differences between the Helsing-Ojala algorithm and our scheme are: (1) our formalism (quadrature rules) allows for “compression” to be performed a priori at the time the quadrature rules are constructed rather than on-the-fly for each specific problem, (2) portions of our scheme (notably the basis construction procedure) generalize trivially to higher dimensions whereas the generalization of the Helsing-Ojala scheme to higher-dimensional problems appears to be difficult, and (3) the Helsing-Ojala scheme is applicable to certain problems with nonsmooth boundary data while the implementation of our scheme assumes smooth boundary data.

This paper is divided into seven sections. After dispensing with preliminary results pertaining to quadrature and interpolation in Section 2, we describe a very general Nyström framework for the discretization of integral equations in Section 3. In Section 4, we review the principal tool of [6], a procedure for the construction of a basis spanning the restriction of solutions of a particular boundary integral equation to a curve segment. The construction of quadratures for the efficient discretization of a single given curve segment is described in Section 5, while Section 6 describes a procedure for the construction of a collection of universal quadratures for a very general class of planar domains with corner points. Finally, numerical results are reported in Section 7.

2. Generalized quadrature and interpolation

In this section, when $X$ is an $n \times m$ matrix, we will denote by $\sigma_j(X)$ its jth largest singular value. Moreover, for $j > \min(n, m)$ we define $\sigma_j(X) = 0$.

2.1. Discretization of square integrable functions. We shall say that a quadrature rule with nodes $x_1, \ldots, x_n \in [a, b]$ and positive weights $w_1, \ldots, w_n$ discretizes a collection of square integrable functions
$f_1, \ldots, f_m$ defined on the interval $[a, b]$ if
\[
\int_a^b f_i(x)f_j(x)\,dx = \sum_{i=1}^n f_i(x_i)f_j(x_i)w_i
\]
holds for all $i = 1, \ldots, m$ and $j = 1, \ldots, m$.

If $x_1, \ldots, x_n, w_1, \ldots, w_n$ is a quadrature discretizing a collection of functions $f_1, \ldots, f_m$ in $L^2([a, b])$, then the map $T$ from the span $S$ of the $f_j$ to the Euclidean space $\mathbb{R}^n$ taking the function $f$ to the vector
\[
\begin{pmatrix}
  f(x_1)\sqrt{w_1} \\
  f(x_2)\sqrt{w_2} \\
  \vdots \\
  f(x_n)\sqrt{w_n}
\end{pmatrix}
\]
is a Hilbert space isomorphism of the subspace $S$ onto the subspace of the Euclidean space $\mathbb{R}^n$ spanned by the vectors
\[
\begin{pmatrix}
  f_1(x_1)\sqrt{w_1} \\
  f_1(x_2)\sqrt{w_2} \\
  \vdots \\
  f_1(x_n)\sqrt{w_n}
\end{pmatrix}, \ldots, \begin{pmatrix}
  f_m(x_1)\sqrt{w_1} \\
  f_m(x_2)\sqrt{w_2} \\
  \vdots \\
  f_m(x_n)\sqrt{w_n}
\end{pmatrix}.
\]

2.2. **Numerical rank.** The numerical rank of a matrix $A$ to precision $\epsilon$ is defined to be the least integer $k$ such that $\sigma_{k+1}(A) \leq \epsilon$. Moreover, we define the numerical rank to precision $\epsilon$ of a collection $f_1, \ldots, f_m$ of square integrable functions on the interval $[a, b]$ to be the numerical rank to precision $\epsilon$ of the matrix
\[
\begin{pmatrix}
  f_1(x_1)\sqrt{w_1} & f_2(x_1)\sqrt{w_1} & \cdots & f_m(x_1)\sqrt{w_1} \\
  f_1(x_2)\sqrt{w_2} & f_2(x_2)\sqrt{w_2} & \cdots & f_m(x_2)\sqrt{w_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1(x_n)\sqrt{w_n} & f_2(x_n)\sqrt{w_n} & \cdots & f_m(x_n)\sqrt{w_n}
\end{pmatrix},
\]
where $x_1, \ldots, x_n, w_1, \ldots, w_n$ is any quadrature discretizing $f_1, \ldots, f_m$.

2.3. **Rank-revealing QR decompositions.** A partial QR decomposition for an $m \times n$ matrix $A$ with $m \leq n$ is a factorization of the form
\[
A = Q \begin{pmatrix}
  R_{11} & R_{12} \\
  0 & R_{22}
\end{pmatrix},
\]
where $Q$ is an $m \times m$ orthogonal matrix, $\Pi$ is an $n \times n$ permutation matrix, $R_{11}$ is a $k \times k$ upper triangular matrix with nonnegative diagonal entries, $R_{12}$ is a $k \times (n-k)$ matrix, and $R_{22}$ is an $(m-k) \times (n-k)$ matrix. For any factorization of this type, we have
\[
\sigma_i(R_{11}) \leq \sigma_i(A) \quad \text{and} \quad \sigma_j(R_{22}) \geq \sigma_{k+j}(A)
\]
for $1 \leq i \leq k$ and $1 \leq j \leq (n-k)$. In [11], it is shown that for any $m \times n$ matrix $A$ with $m \leq n$ and any integer $1 \leq k \leq m$, there exists a factorization of this form such that $\|R_{11}^{-1}R_{12}\|_\infty \leq d(k, n)$,
\[
\sigma_i(A) \leq c(k, n) \sigma_i(R_{11})
\]
holds for $i = 1, \ldots, k$, and
\[
c(k, n) \sigma_{k+j}(A) \geq \sigma_j(R_{22})
\]
holds for $j = 1, \ldots, n-k$, where $c(k, n) = \sqrt{1+k(n-k)}$ and $d(k, n) = 1$. Moreover, [11] gives a stable algorithm for the computation of partial QR factorizations satisfying bounds of this type with $c = \sqrt{1+nk(n-k)}$ and $d = \sqrt{n}$ which requires $O(m^2n)$ floating point operations — that is, the same asymptotic complexity as the well-known pivoted Gram-Schmidt algorithm$^2$.

$^2$In fact, the performance of the scheme described in [11] depends on a parameter which affects both its running time and the resulting bounds. The values reported here represent but one possible configuration.
Remark 2.1. Our implementation of the algorithm described in this paper uses the pivoted Gram-Schmidt procedure with reorthogonalization in place of the algorithm of [11] to compute rank-revealing QR factorizations. As is observed in [11] and discussed thoroughly in the monograph [4], the pivoted Gram-Schmidt algorithm with reorthogonalization, which is easier to implement and somewhat faster than the algorithm of [11], works well in practice despite the existence of counterexamples which show that it can fail in certain circumstances.

2.4. Generalized Chebyshev quadratures. A quadrature formula will be referred to as a Chebyshev quadrature for a set of $2n$ linearly independent functions $\phi_1, \ldots, \phi_{2n} : [a, b] \to \mathbb{R}$ if it consists of $2n$ nodes and $2n$ weights and integrates $\phi_i$, for all $i = 1, \ldots, 2n$.

The construction of a Chebyshev quadrature for an orthonormal collection of square integrable functions $u_1, \ldots, u_k$ defined on an interval $[a, b]$ is trivial given a preexisting quadrature $x_1, \ldots, x_n, w_1, \ldots, w_n$ integrating products of those functions. Let

$$U = \begin{pmatrix} u_1(x_1)\sqrt{w_1} & u_1(x_2)\sqrt{w_2} & \cdots & u_1(x_n)\sqrt{w_n} \\ u_2(x_1)\sqrt{w_1} & u_2(x_2)\sqrt{w_2} & \cdots & u_2(x_n)\sqrt{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_k(x_1)\sqrt{w_1} & u_k(x_2)\sqrt{w_2} & \cdots & u_k(x_n)\sqrt{w_n} \end{pmatrix},$$

and

$$r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix},$$

where $r_i, i = 1, \ldots, k$, is defined by

$$r_i = \int_a^b u_i(x) \, dx.$$

By virtue of the orthonormality of the $u_1, \ldots, u_k$ and the requirement that the quadrature integrates products of the $u_i$, the rows of the matrix $U$ are orthonormal. It follows that the matrix $U$ has $k$ nonzero singular values, all of which are 1. The results of [11] discussed in Section 2.3 now imply that a decomposition of $U$ of the form

$$U \Pi = Q \begin{pmatrix} R_{11} & R_{12} \end{pmatrix},$$

where $Q$ is an orthogonal $k \times k$ matrix, $R_{12}$ is an $k \times (n-k)$ matrix and $R_{11}$ is a $k \times k$ matrix such that

$$\frac{1}{\sqrt{1 + nk(n-k)}} \leq \sigma_j(R_{11}) \leq 1 \quad (2.3)$$

for $1 \leq j \leq k$, can be computed stably in at most $O(n^3)$ operations. A vector $z$ with at most $k$ nonzero entries such that

$$U z = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix},$$

can be computed by solving the $k \times k$ linear system

$$R_{11} \hat{z} = Q^* r, \quad (2.4)$$

the condition number of which is bounded by

$$\sqrt{1 + nk(n-k)}$$
by virtue of the inequality (2.3), and setting \(z\) to be
\[
\Pi^{-1} \begin{pmatrix} \tilde{z} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
If \(i_1, \ldots, i_k\) denote the indices of the nonzero components of \(z\) and we let
\[
y_j = x_{i_j} \quad \text{and} \quad v_j = z_{i_j} \sqrt{w_{i_j}}
\]
for \(j = 1, \ldots, k\), then we have
\[
\begin{pmatrix}
\begin{bmatrix}
  u_1(y_1) & u_1(y_2) & \cdots & u_1(y_k)
  \\
  u_2(y_1) & u_2(y_2) & \cdots & u_2(y_k)
  \\
  \vdots & \vdots & \ddots & \vdots
  \\
  u_k(y_1) & u_k(y_2) & \cdots & u_k(y_k)
\end{bmatrix}

\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_k
\end{bmatrix}
\end{pmatrix}
= r;
\]
that is, \(y_1, \ldots, y_k, v_1, \ldots, v_k\) is a Chebyshev quadrature for \(u_1, \ldots, u_k\).

2.5. Chebyshev quadratures and interpolation. If \(f\) is a linear combination
\[
f(x) = \sum_{j=1}^{k} \alpha_j u_j(x) \tag{2.5}
\]
of the orthonormal functions \(u_1, \ldots, u_k\) in \(L^2[a, b]\) and \(x_1, \ldots, x_n, w_1, \ldots, w_n\) is a quadrature integrating products of the \(u_j\), then the coefficients \(\alpha_1, \ldots, \alpha_k\) in the expansion (2.5) can be computed stably from the values of the function \(f\) at the quadrature nodes \(x_1, \ldots, x_n\). In particular, let \(U\) be the \(n \times k\) matrix whose entries are given by
\[
U_{ij} = u_j(x_i) \sqrt{w_i},
\]
let
\[
F = \begin{pmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_n)
\end{pmatrix},
\]
and denote by \(W\) the \(n \times n\) diagonal matrix with entries
\[
W_{ii} = \sqrt{w_{ii}}.
\]
Then
\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{pmatrix}
= U^*WF.
\]
Note that \(UU^*(WF) = WF\) since \(WF\) is in the span of the column space of \(U\) by assumption. The computation of \(\alpha_1, \ldots, \alpha_k\) in this fashion is entirely numerically stable since the rows of \(U^*\) are orthonormal and \(W\) is a diagonal matrix. If a scheme is available for evaluating the functions \(u_j\) at arbitrary points \(x\), then this mechanism provides a stable means for interpolating functions in the span of \(u_1, \ldots, u_k\) from \(x_1, \ldots, x_n\) to arbitrary points.

In fact, interpolation from the nodes of a Chebyshev quadrature for \(u_1, \ldots, u_k\) constructed in the manner described in the preceding section is also stable. To see this, suppose \(x_{i_1}, \ldots, x_{i_k}\) are the nodes
of such a quadrature. Then, by virtue of the inequality (2.3) appearing in the preceding section, the condition number of the matrix $\tilde{U}$ defined by

$$
\tilde{U} = \begin{pmatrix}
  u_1(x_{i_1})\sqrt{w_{i_1}} & u_2(x_{i_1})\sqrt{w_{i_1}} & \cdots & u_k(x_{i_1})\sqrt{w_{i_1}} \\
  u_1(x_{i_2})\sqrt{w_{i_2}} & u_2(x_{i_2})\sqrt{w_{i_2}} & \cdots & u_k(x_{i_2})\sqrt{w_{i_2}} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_1(x_{i_k})\sqrt{w_{i_k}} & u_2(x_{i_k})\sqrt{w_{i_k}} & \cdots & u_k(x_{i_k})\sqrt{w_{i_k}}
\end{pmatrix}
$$

is bounded by $\sqrt{1 + nk(n-k)}$. By construction, the vector

$$
\begin{pmatrix}
  f(x_{i_1})\sqrt{w_{i_1}} \\
  f(x_{i_2})\sqrt{w_{i_2}} \\
  \vdots \\
  f(x_{i_k})\sqrt{w_{i_k}}
\end{pmatrix}
$$

is in the span of the columns of $\tilde{U}$, so it follows that the linear system

$$
\tilde{U}\begin{pmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_k
\end{pmatrix} = \begin{pmatrix}
  f(x_{i_1})\sqrt{w_{i_1}} \\
  f(x_{i_2})\sqrt{w_{i_2}} \\
  \vdots \\
  f(x_{i_k})\sqrt{w_{i_k}}
\end{pmatrix}
$$

admits a unique solution which can be stably computed.

3. **Nyström Discretization**

The Nyström discretization of the integral equation

$$
\lambda \sigma(x) + \int_{\Gamma} K(x, y) \sigma(y) dS(y) = u(x)
$$

(3.1)

proceeds by subdividing the curve $\Gamma$ into segments, assuming the solution $\sigma$ admits a representation with respect to a local basis on each segment, and approximating the integral in Equation (3.1) via quadrature formulae integrating the products of these basis functions with the integral kernel $K(x, y)$. See [2] for a detailed discussion of the Nyström discretization of boundary integral equations.

We now describe a very general Nyström framework for the discretization of Laplace and Helmholtz boundary integral equations of the form (3.1). We begin by assuming that $\Gamma$ is divided into $n$ curve segments, $\Gamma_1, \ldots, \Gamma_n$, not necessarily of equal length. For $j = 1, \ldots, n$, let $B_j$ denote a ball of minimum radius containing $\Gamma_j$. For each curve segment $\Gamma_i$ we will require the following:

1. An orthonormal collection of $n_i$ basis functions $\phi^{i,1}_1, \ldots, \phi^{i,n_i}_1$ in $L^2(\Gamma_i)$;
2. A collection of interpolation nodes $\lambda^{i,1}_1, \ldots, \lambda^{i,n_i}_1$ and a scheme for interpolating the basis functions $\phi^{i,1}_1, \ldots, \phi^{i,n_i}_1$ from nodes $\lambda^{i,1}_1, \ldots, \lambda^{i,n_i}_1$ to their values at arbitrary points;
3. A “far” quadrature formula accurate for

$$
\int_{\Gamma_i} K(x, y) \phi^{i,j}_1(y) dS(y), \quad j = 1, \ldots, n_i,
$$

(3.2)

whenever $x$ is a point in the intersection of $\Gamma$ and the complement of $2B_j$;
4. A “near” quadrature formula for integrals of the form

$$
\int_{\Gamma_i} K(x, y) \phi^{i,j}_1(y) dS(y), \quad j = 1, \ldots, n_i,
$$

which holds for all $x$ in the intersection of the annulus $2B_j \setminus B_j$ and $\Gamma$;
5. A set of $n_i$ “diagonal” quadrature formulas, one for each of the interpolation nodes $\lambda^{i}_j$, each of which is of the form

$$
\int_{\Gamma_i} K(\lambda^{i}_j, y) \sigma(y) dS(y) \approx \sum_i K(\lambda^{i}_j, y_i) \sigma(y_i) w_i
$$
and holds whenever $\sigma$ is in the span of $\phi_1^l, \ldots, \phi_{n_l}^l$.

**Remark 3.1.** In the case of smooth curve segments, the requisite quadratures for the potential theoretic kernels considered in this paper are easy to generate. Suppose that $\Gamma_l$ is parameterized by $r: [-1,1] \rightarrow \Gamma_1$. Since solutions of the integral equations are smooth on $\Gamma_1$, an appropriate local basis can be obtained by mapping the Legendre polynomials of a given order $k$ onto $\Gamma_1$ via $r$. Similarly, the image of the Legendre quadrature nodes of order $k$ under $r$ can serve as interpolation nodes $\lambda_1^l, \ldots, \lambda_k^l$.

For $x$ sufficiently separated from $y$, the potential theoretic kernels under consideration are smooth (in fact, the restriction of $K(x,y)$ to the region $\{(x,y) \mid x \in (2B_l)^c \cap \Gamma, y \in B_l\}$ admits representation as a multipole or partial wave expansion with exponentially decaying coefficients). It follows that the image of a Legendre quadrature under $r$ can be used as a “far” quadrature. If $K(x,y)$ is smooth, Legendre quadratures can also serve as “near” and “diagonal” quadratures. However, in some boundary integral formulations studied in this paper, the kernel $K(x,y)$ behaves as $\log(|x-y|)$ for $x \approx y$. In this case, efficient quadrature for integrals of the form

$$
\int_{-1}^{1} \log |s-t|p(t)dt
$$

where $p$ is a Legendre polynomial of a given degree and $s$ is a point in $[-2,-1] \cup [1,2]$ can be easily constructed using the approach of [5]. The mapping of such a quadrature onto $\Gamma_1$ via the parameterization $r$ can be used as a “near” quadrature. Similarly, diagonal quadratures can be constructed by mapping quadratures for integrals of the form

$$
\int_{-1}^{1} \log |\alpha_j-t|p(t)dt,
$$

where $p$ is a Legendre polynomial and $\alpha_j$ is one of the $k$th order Legendre nodes on $[-1,1]$.

The method proceeds under the assumption that the restriction of the unknown solution $\sigma$ in equation (3.1) to the curve segment $\Gamma_l$ can be represented as linear combinations of the basis functions $\phi_1^l, \ldots, \phi_{n_l}^l$ for $\Gamma_l$. Let $\Gamma_l$ and $\Gamma_m$ be two curve segments, not necessarily distinct. We will denote by $s_1, \ldots, s_{n_l}$ the interpolation nodes on the curve segment $\Gamma_l$, and by $t_1, \ldots, t_{n_m}$ the interpolation nodes on the curve segment $\Gamma_m$. The integral equation

$$
T_{ml}\sigma(x) = u(x),
$$

(3.3)

where $T_{ml}$ is the integral operator mapping functions on $\Gamma_m$ to functions on $\Gamma_l$ defined by

$$
T_{ml}\sigma(x) = \int_{\Gamma_l} K(x,y)\sigma(y)dy
$$

(3.4)

is then discretized by repeating the following sequence of steps for each interpolation node $t$ on $\Gamma_m$:

1. The appropriate quadrature formula $x_1, \ldots, x_q, w_1, \ldots, w_q$ for functions of the form

$$
K(t,s)\phi_u^l(s), \quad u = 1, \ldots, n_l,
$$

is determined. That is, depending on the location of $t$ relative to the curve segment $\Gamma_l$, either the “far” quadrature rule, the “near” quadrature rule, or one of the diagonal quadrature rules is selected.

2. The kernel $K(x,y)$ is evaluated at the points $(t, x_r)$ for $r = 1, \ldots, q$ and the $1 \times q$ vector $v$ with entries

$$
v_r = K(t, x_r)w_r
$$

is formed.

3. The $1 \times q$ vector $v$ is multiplied on the right by the $q \times n_l$ matrix interpolating the basis functions $\phi_1^l, \ldots, \phi_{n_l}^l$ from the interpolation nodes $s_1, \ldots, s_{n_l}$ on $\Gamma_l$ to the quadrature nodes $x_1, \ldots, x_q$.

4. The entries $\{\alpha_u\}$ of the resulting $1 \times n_l$ vector give a single linear equation

$$
\alpha_1\sigma(s_1) + \alpha_2\sigma(s_2) + \ldots + \alpha_{n_l}\sigma(s_{n_l}) = u(t)
$$

constraining the values of the solution $\sigma$ at the nodes $s_1, \ldots, s_{n_l}$. 


The result of repeating this procedure for each of the $n_m$ interpolation nodes is an $n_m \times n_l$ linear system of the form

$$
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n_l} \\
    a_{21} & a_{22} & \ldots & a_{2n_l} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n_m1} & a_{n_m2} & \ldots & a_{n_mn_l}
\end{pmatrix}
\begin{pmatrix}
    \sigma(s_1) \\
    \sigma(s_2) \\
    \vdots \\
    \sigma(s_{n_l})
\end{pmatrix}
= 
\begin{pmatrix}
    u(t_1) \\
    u(t_2) \\
    \vdots \\
    u(t_{n_m})
\end{pmatrix}
$$

discretizing the integral equation

$$
T_{ml}\sigma = u.
$$

Repeating the above procedure for each pair of curve segments $\Gamma_l$ and $\Gamma_m$, and accounting for the constant term in equation (3.1), results in a discrete system of $N$ equations in $N$ unknowns of the form

$$
\lambda x + Ax = y,
$$

where $N$ is equal to the sum of the number of interpolation nodes $n_j$ on each curve segment $\Gamma_l$ over $l = 1, \ldots, n$ and $A$ is a matrix formed by concatenating the discrete matrices representing the $T_{ml}$.

Solving the amalgamated system yields the values of the unknown function $\sigma$ in (3.1) at the interpolation nodes of each of the curve segments $\Gamma_l$. The value of $\sigma$ at any point $x$ on $\Gamma$ can then be computed in $O(1)$ operations using the appropriate interpolation formula. Moreover, the value of a layer potential

$$
u(x) = \int_{\Gamma} K(x,y)\sigma(y)\,dy$$

can be computed for any $x$ sufficiently far enough away from the curve $\Gamma$ in $O(n)$ operation using the far quadrature formulas for the curve segments $\Gamma_l$, $l = 1, \ldots, n$. For points close to the curve, an adaptive Gaussian quadrature scheme which relies on the ability to evaluate the charge distribution at any point via interpolation can be used to compute the value of the layer potential.

4. Charge bases

In this section, we introduce the principal analytical tool of this paper. Specifically, we show that under mild assumptions a small finite orthonormal basis spanning the restrictions of solutions of a Laplace or Helmholtz boundary integral equation on a contour $\Gamma$ to a small curve segment $\Gamma_0 \subset \Gamma$ can be constructed. We will refer to such a basis as a “charge basis” for the contour $\Gamma_0$ (the underlying integral equation will always be apparent from context).

In the interests of clarity and brevity, we will restrict our attention here to the boundary integral equation

$$
-\frac{1}{2} \sigma(x) + \int_{\Gamma} \left( \frac{i}{4} \nu(y) \cdot \nabla y + 1 \right) H_0(k |x - y|) \sigma(y) \, dS(y) = u(x),
$$

(4.1)

where $H_0$ is the Hankel function of zeroth order and $\nu(y)$ denotes the outward unit normal vector on the contour $\Gamma$. The equation (4.1) can be used to obtain a solution of the exterior Dirichlet boundary value problem for the Helmholtz equation with wavenumber $k$ (see, for instance, [9]). The argument given here clearly applies to a variety of other boundary integral equations with minor modifications.

4.1. Partial wave expansions. If the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfies the Helmholtz equation

$$
(\nabla^2 + k^2) \phi = 0
$$

in a disc $D$ of radius $R > 0$ centered at $x_0$ and the radiation condition

$$
\lim_{t \to \infty} \phi(tx) e^{-ikt|x|} = O \left( t^{-1/2} \right)
$$

uniformly at infinity, then $\phi$ can be represented uniquely as a $J$-expansion in $D$; i.e., in the form

$$
\phi(x) = \sum_{m=-\infty}^{\infty} a_m J_m(kr)e^{im\theta},
$$

(4.2)
where \((r, \theta)\) denote the usual polar coordinates centered at \(x_0\) and \(J_m\) is the Bessel function of the first kind of order \(m\). The following theorem, which appears in [14], establishes the convergence rate of the expansion (4.2).

**THEOREM 4.1.** If \(D_1 \subset D\) is a disc of radius \(R_1 < R\) centered at \(x_0\), then there exists a constant \(c > 0\) such that for \(x \in D_1\) and \(N > |k| R_1\),

\[
|\phi(x) - \sum_{m=-N}^{N} \alpha_m J_m(kr)e^{im\theta}| < c \left( \frac{R_1}{R} \right)^N.
\]

**4.2. Rank of interaction.** If \(\Gamma_0\) and \(\Gamma_1\) are two disjoint compact piecewise smooth curves in the plane, then the mapping defined by

\[
Tf(x) = \int_{\Gamma_0} K(x,y)f(y)dS(y),
\]

where \(K(x,y)\) is the kernel appearing in (4.1), is compact as an operator \(L^2(\Gamma_0) \rightarrow L^2(\Gamma_1)\). This follows, for example, from the fact that \(K(x,y)\) can be written as

\[
K(x,y) = f(x,y) + g(x,y) \log |x - y|, \quad (4.4)
\]

where \(f(x,y)\) and \(g(x,y)\) are smooth (see [1]). The identity (4.4) implies that \(T\) is a Hilbert-Schmidt integral operator; i.e., \(K(x,y)\) is in \(L^2(\Gamma_1 \times \Gamma_0)\). As a compact operator, \(T\) admits a singular value decomposition and we shall let \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\) denote the singular values of \(T\). If the rank of \(T\) is finite, we will adopt the convention that \(\lambda_j = 0\) for all \(j\) greater than the rank of \(T\). With this convention, we ensure that \(\lambda_j \to 0\) as \(j \to \infty\).

**DEFINITION 4.1.** We will call the least integer \(m\) such that \(\lambda_{m+1} < \epsilon\) the \(\epsilon\)-rank of interaction of the curve segments \(\Gamma_0\) and \(\Gamma_1\) under the kernel \(K\).

![Figure 1: The \(\epsilon\)-rank of interaction of the curve segments \(\Omega_0 \cup \Omega_1\) and \(\Omega_2 \cup \Omega_3\) behaves as \(\log^2(\epsilon)\).](image)

If the \(\epsilon\)-rank of interaction of the operator (4.3) is \(m\), then there exists a quadrature formula of the form

\[
\int_{\Gamma_1} K(x,y)f(y)dS(y) \approx \sum_{j=1}^{m} K(x,y_j)f(y_j)w_j,
\]

with the \(y_j\) lying in \(\Gamma_1\), which holds with precision approximately \(\epsilon\) for \(x \in \Gamma_0\) (note that the kernels considered in this paper are bounded almost everywhere for \(x \in \Gamma_0\) and \(y \in \Gamma_1\)). The existence of such quadratures follows readily from the discussion of generalized Chebyshev quadratures in Section 2. Although the rank of interaction between two curve segments under one of the potential-theoretic kernels considered here can be high, for most curves appearing in practice the singular values of the integral
operators corresponding to interactions between neighboring curve segments decay exponentially (see, for instance, [10], which contains several relevant examples). In the particular case of corner regions, which are the focus of this paper, the rank of interaction between a neighborhood of a corner and the neighboring portions of the curve is typically low regardless of the angle of the corner.

Estimates of the decay of singular values of operators of the form \( T \) as a function of the geometry of the curve are a topic of some interest; however, a detailed discussion of such estimates is well beyond the scope of this paper. We will content ourselves with a rank of interaction estimate in a simple case. Specifically, we will consider the operator \( S \) defined by

\[
Sf(t) = \int_{-1}^{1} \log |r(t) - r(s)| f(s) |r'(s)| \, ds,
\]

where \( r : [-2, 2] \to \mathbb{R}^2 \) is defined by \( r(t) = \{x(t), y(t)\} \) with

\[
x(t) = \begin{cases} \cos(\theta)t & t > 0 \\ -\cos(\theta)t & t < 0 \end{cases}
\]

and \( y(t) = \sin(\theta)t \) for an angle \( \theta \). We will let \( \Omega_0 \) denote the curve \( \{r(t) \mid 0 \leq t < 1\} \), \( \Omega_1 \) denote the curve \( \{r(t) \mid -1 \leq t < 0\} \), \( \Omega_2 \) denote the curve \( \{r(t) \mid 1 \leq t \leq 2\} \), and \( \Omega_3 \) denote the curve \( \{r(t) \mid -2 \leq t < -1\} \). The curve segments are depicted in Figure 1. We will show that the \( \epsilon \)-rank of interaction of the curve segments \( \Omega_0 \cup \Omega_1 \) and \( \Omega_2 \cup \Omega_3 \) under the kernel \( \log |x - y| \) behaves as \( O(\log^2(\epsilon)) \). The kernel \( \log |x - y| \) was chosen to simplify the proof and notation; the same arguments clearly apply to the kernel \( K(x, y) \).

The \( \epsilon \)-rank of interaction of \( \Omega_0 \cup \Omega_1 \) and \( \Omega_2 \cup \Omega_3 \) is dominated by the interactions of \( \Omega_0 \) with \( \Omega_2 \) and \( \Omega_1 \) with \( \Omega_3 \); i.e., since the curve segments involved are separated, the “cross” interactions of \( \Omega_0 \) with \( \Omega_3 \) and \( \Omega_1 \) with \( \Omega_2 \) are of lower rank compared with those of \( \Omega_0 \) with \( \Omega_2 \) and \( \Omega_1 \) with \( \Omega_3 \). The \( \epsilon \)-ranks of the interaction of \( \Omega_0 \) with \( \Omega_2 \) and \( \Omega_1 \) with \( \Omega_3 \) can be estimated quite easily. First, we observe that the rank of interaction is invariant under rotation and scaling. This allows us to reduce the problem to the determination of the \( \epsilon \)-rank of operator \( L^2([-1, 0]) \to L^2([0, 1]) \) defined by

\[
\tilde{S}f(x) = \int_{-1}^{1} \log |x - y| f(y) \, dy.
\]

A proof that the \( \epsilon \)-rank of the operator \( \tilde{S} \) behaves as \( O(\log^2(\epsilon)) \) appears in the appendix to this paper. It follows from that proof and this discussion that the \( \epsilon \)-rank of interaction of the curve segments \( \Omega_0 \cup \Omega_1 \) and \( \Omega_2 \cup \Omega_3 \) behaves as \( \log^2(\epsilon) \).

4.3. Existence of charge bases. In what follows, we shall fix a simply-connected domain \( \Omega \) in the plane whose boundary \( \Gamma \) is a compact piecewise smooth curve and let \( \Gamma_0 \subset \Gamma \) be a curve segment.

Our aim is to show that under these assumptions there exists a small orthonormal basis approximately spanning the set of solutions of the boundary integral equation (4.1) to a small curve segment \( \Gamma_0 \subset \Gamma \) under a mild assumption on boundary data \( u(x) \).

We will denote by \( K(x, y) \) the integral kernel

\[
\left( \frac{i}{4} \nu(y) \cdot \nabla_y + 1 \right) H_0(k |x - y|)
\]

appearing in equation (4.1). Moreover, shall let \( B \) be a disc of minimum radius which contains \( \Gamma_0 \) and we will denote by \( \Gamma_1 \) the portion of the contour \( \Gamma \) contained in \( 2B \setminus \Gamma \). We will let \( \Gamma_2 \) denote the portion of the curve \( \Gamma \) contained in the complement of the ball \( 2B \). Figure 2 depicts the situation. Finally, we will assume that the right-hand side \( u(x) \) satisfies the Helmholtz equation in the ball \( 2B \) and the radiation condition at infinity.

The boundary integral equation (4.1) can be rearranged as

\[
-\frac{1}{2} \sigma(x) + \int_{\Gamma_0} K(x, y)\sigma(y)ds(y) = u(x) - \int_{\Gamma_1} K(x, y)\sigma(y)ds(y) - \int_{\Gamma_2} K(x, y)\sigma(y)ds(y) - \int_{\Gamma_1} K(x, y)\sigma(y)ds(y). \tag{4.5}
\]
By virtue of Theorem 4.1 and our assumption on the right-hand side $u(x)$, we can introduce the approximation

$$u(x) - \int_{\Gamma_2} K(x, y) \sigma(y) dS(y) \approx \sum_{m=-N}^{N} \alpha_m J_m(kr)e^{im\theta},$$

which holds for $x \in \Gamma_0$; i.e., both the integral over $\Gamma_2$ and the right-hand side $u(x)$ admit representation as partial wave expansions by Theorem 4.1. Moreover, as described in Section 4.2, when $x \in \Gamma_0$ the third term on the right hand side of equation (4.5) can be approximated as

$$\int_{\Gamma_1} K(x, y)f(y)dS(y) \approx \sum_{j=1}^{M} K(x, y_j)f(y_j)w_j,$$

where the $y_j$ lie in $\Gamma_1$ and $M$ is the rank of interaction of $\Gamma_0$ with $\Gamma_1$. It follows that the restriction of $\sigma$ to the curve segment $\Gamma_0$ satisfies the integral equation

$$-\frac{1}{2} \sigma(x) + \int_{\Gamma_0} K(x, y) \sigma(y) dS(y) = \sum_{m=-N}^{N} \alpha_m J_m(kr)e^{im\theta} - \sum_{j=1}^{M} K(x, y_j)f(y_j)w_j.$$

It is now clear how to form a charge basis for $\Gamma_0$. We observe that the restriction of (4.1) to $\Gamma_0$ is invertible as an operator $L^2(\Gamma_0) \to L^2(\Gamma_0)$ and form the collection of functions obtained by solving the restricted integral equation for each of the functions of the form

$$J_m(kr)e^{im\theta} \quad \text{and} \quad K(x, y_j)$$

appearing in (4.7). The resulting functions are orthonormalized in order to form a charge basis.

4.4. Convergence estimate. In this section, we derive a crude estimate on the number of charge basis functions required to achieve a given $L^\infty(\Gamma_0)$ precision $\epsilon$ for the approximation (4.7). And of course, this quantity is related to the $L^2(\Gamma_0)$ error in the approximation of restricted solutions by the charge basis through the $L^2(\Gamma_0)$ norm of the inverse of the restriction of the integral operator (4.1) and the arclength of the curve $\Gamma_0$.

In particular, if $N$ is chosen so that

$$N \geq \max \left( |k| R_1, -\frac{\log (\epsilon) + \log (c)}{\log (2)} \right),$$

where $R_1$ is the radius of $B$ and $c$ is the constant appearing in Theorem 4.1, then the approximation (4.6) is guaranteed to achieve precision $\epsilon$. The rank of interaction $M$ of $\Gamma_0$ and $\Gamma_1$ depends on the geometry.
of the contour $\Gamma$ and the wavenumber $k$; however, in typical cases — especially when $\Gamma_0$ is taken to be a small region around a corner point — $M$ behaves as $\log^2(\epsilon)$. Thus the number of charge basis functions required for precision $\epsilon$ is roughly
\[
\max \left( |k| R_1, O \left( \log^2(\epsilon) \right) \right).
\]
In other words, the convergence of approximations with respect to charge bases is exponential once a certain number of “terms per wavelength” has been achieved. This means that the number of basis functions necessary to represent a charge distribution on many types of complicated curve segments (e.g. corner regions) is comparable with the number required to represent a charge distribution on a smooth curve segment using typical approaches (e.g., piecewise Gaussian polynomials). This estimate is consistent with our numerical experiments, which are presented in Section 7.

Remark 4.1. This estimate only provides a very crude upper bound for the dimension of the resulting charge basis; charge bases are almost always much smaller than this estimate suggests. This is partly a consequence of the fact that $O \left( \log^2(\epsilon) \right)$ is a loose upper bound for the rank of interaction and partly because the subspaces formed by inverting the $J$-functions overlap with those formed by inverting the kernel functions $K(x_i, y)$.

Remark 4.2. In the case of the boundary integrals related to Laplace’s equation, considerations involving the wavenumber are (obviously) unnecessary. For those integral equations, in typical cases the number of charge basis functions behaves as $O(\log^2(\epsilon))$.

5. Purpose-made quadratures

In this section, we describe a numerical algorithm for the construction of a charge basis $\sigma_1, \ldots, \sigma_k$ for a Laplace or Helmholtz boundary integral equation
\[
\lambda \sigma(x) + \int_{\Gamma} K(x, y) \sigma(y) \, dS(y) = u(x) \quad (x \in \Gamma_0)
\]
over a corner curve segment $\Gamma_0 \subset \Gamma$ as well as an algorithm for the computation of purpose-made quadratures for its discretization over $\Gamma_0$ given a charge basis for $\Gamma_0$. In what follows $\Gamma_1$ will denote a small portion of the curve $\Gamma$ neighboring $\Gamma_0$.

5.1. Numerical construction of a charge basis. The algorithm of this section takes as input a precision $\epsilon$, a parameterization $r : [-1, 1] \rightarrow \Gamma_0$ that maps 0 to the corner point of $\Gamma_0$, two integer parameters $s$ and $k$ whose roles will be described below, and a quadrature of the form
\[
\int_{\Gamma_1} K(x, y) \sigma(y) \, dS(y) \approx \sum_{j=1}^{N} K(x, z_j) \sigma(z_j) w_j
\]
which holds for points $x$ in $\Gamma_0$.

It proceeds as follows:

1. Form a simply-graded mesh\(^1\) on the corner curve segment $\Gamma_0$ by constructing the subintervals of $[-1, 1]$ with endpoints
\[
\frac{1}{2j} \quad \text{and} \quad -\frac{1}{2j} \quad \text{for} \quad j = 1, 2, \ldots, s,
\]
and taking the image of the resulting subintervals under the parameterization $r$.

2. Discretize the restricted boundary integral equation
\[
\lambda \sigma(x) + \int_{\Gamma_0} K(x, y) \sigma(y) \, dS(y) = u(x) \quad (x \in \Gamma_0)
\]
via the Nyström method under the assumption that the solution $\sigma$ can be represented as a $k$th order piecewise Legendre expansion over the simply-graded mesh formed in Step 1. Denote by $x_1, \ldots, x_m, w_1, \ldots, w_m$ the nodes and weights of the piecewise Legendre quadrature which represents solutions of (5.3) on $\Gamma_0$.

\(^1\)The terminology is adapted from [13]. Also see [6].
3. Solve the discrete linear system formed in Step 2 for a collection of right-hand-sides consisting of multipoles and functions of the form

\[ K(x, z_j), \]

where the \( z_1, \ldots, z_N \in \Gamma_1 \) are points in of the quadrature formula (5.2). The order of the multipole functions is chosen so as to ensure very high accuracy in the approximation of potential functions on \( \Gamma_0 \).

Denote the resulting functions by \( u_1, \ldots, u_n \). Note that the values of the \( u_j \) at arbitrary points on \( \Gamma_0 \) can be calculated using standard interpolation techniques for piecewise Legendre polynomials.

4. Form the \( m \times n \) matrix \( A \) whose entries are given by

\[ A_{ij} = u_j(x_i) \sqrt{w_i}. \]

5. Form a rank-revealing QR decomposition

\[ A\Pi = Q \left( \begin{array}{cc} R_{11} & R_{12} \\ 0 & R_{22} \end{array} \right) \]

for \( A \) and let \( k \) be the least integer such that the \( k \)th singular value of \( A \) is less than the input precision \( \epsilon \).

6. Define the charge basis functions \( \sigma_1, \ldots, \sigma_k \) via the formula

\[ \sigma_j(x_i) = Q_{ij} / \sqrt{w_i}, \]

where \( Q_{ij} \) refers to the \( ij \)th entry of the matrix \( Q \) formed in the preceding step. Note that since the values of the \( \sigma_j \) are known at the points of a piecewise Legendre quadrature, their values at any point on \( \Gamma_0 \) can be readily obtained via interpolation.

The parameters \( s \) and \( k \) should be chosen so as to ensure that the piecewise Legendre quadrature used to discretize solutions of the restricted boundary integral equation (5.3) integrates products of the solutions obtained in Step 3.

5.2. Numerical construction of the quadrature formulae. In this section, we detail the construction of the quadrature formulae for the discretization of the boundary integral equation (5.1) over \( \Gamma_0 \).

The algorithm of this section takes as input the quadrature \( x_1, \ldots, x_m, w_1, \ldots, w_m \) and the charge basis \( \sigma_1, \ldots, \sigma_k \) formed using the procedure of the preceding section as well as an integer parameter \( k_{\text{lege}} \). We may, by virtue of the parameterization \( r : [-1, 1] \rightarrow \Gamma_0 \) regard the functions \( \sigma_j \) as given over the interval \([-1, 1]\).

The algorithm proceeds as follows:

1. Construct a Chebyshev quadrature for the charge basis functions \( \sigma_1, \ldots, \sigma_k \). The nodes \( \lambda_1, \ldots, \lambda_k \) of this quadrature serve as discretization nodes for Nyström procedure of Section 3. Note that the nodes \( \lambda_1, \ldots, \lambda_k \) are guaranteed to be stable interpolation nodes for the charge basis functions (as described in Section 2.5).

2. Form the “far” quadrature required by the Nyström procedure by constructing a Chebyshev quadrature for functions of the form

\[ \sigma_j(x) (p(x) + q(x)), \]

where \( 1 \leq j \leq k, p(x) \) is a polynomial of degree \( k_{\text{lege}} \) on the interval \([-1, 0]\), and \( q(x) \) is a polynomial of degree \( k_{\text{lege}} \) on the interval \([0, 1]\).

3. Construct the “near” quadrature by constructing a Chebyshev quadrature for functions of the form

\[ \log |t - x| \sigma_j(x), \]

where \( j = 1, \ldots, k \) and \( t \) is a point on \( \Gamma \) close to \( \Gamma_0 \).
4. For each \( i = 1, \ldots, k \), construct a diagonal quadrature for functions of the form
\[
K(\lambda_i, x)\sigma_j(x),
\]
\( j = 1, \ldots, k \).

5. For each of the \( k + 2 \) quadratures constructed in the last section, form the matrix interpolating the charge basis functions from the discretization nodes \( \lambda_1, \ldots, \lambda_k \) to the quadrature nodes. This can be achieved via the procedure described in Section 2.5.

6. Universal quadratures for domains with corners

We now indicate how the procedure of the preceding section can be modified in order to produce a collection of quadrature formulae suitable for the discretization of a very general class of domains with corner points. Once again we shall restrict our discussion to the integral equation (4.1) for the sake of brevity.

We shall say that a contour \( \gamma \) is a corner region with corner point at \((x_0, y_0)\) if it admits a parameterization \( r(t) = (x(t), y(t)) \) such that
\[
x(t) = \begin{cases} x_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots & \text{for } a < t < 0 \\
                      x_0 + b_1 t + b_2 t^2 + b_3 t^3 + \ldots & \text{for } 0 < t < b \\
                      x_0 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots & \text{for } a < t < 0 \\
                     y_0 + d_1 t + d_2 t^2 + d_3 t^3 + \ldots & \text{for } 0 < t < b,
\end{cases}
\]
(6.1)

where the series converge uniformly for \( t \neq 0 \), and \((a_1, c_1) \neq (b_1, d_1)\). We can assume without loss of generality that (6.1) is an arclength parameterization; i.e., that \( x'(t)^2 + y'(t)^2 = 1 \). For a contour of this type, we define the oriented angle \( \theta \) of the corner as the angle of the counter-clockwise rotation required to transform the vector
\[
\lim_{t \to 0^-} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix}
\]
into the vector
\[
\lim_{t \to 0^+} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} b_1 \\ d_1 \end{pmatrix}.
\]
Furthermore, we define the left and right instantaneous curvatures by
\[
\kappa_- = \lim_{t \to 0^-} \left[ x'(t)y''(t) - y'(t)x''(t) \right] = 2(a_1c_2 - c_1a_2)
\]
\[
\kappa_+ = \lim_{t \to 0^+} \left[ x'(t)y''(t) - y'(t)x''(t) \right] = 2(b_1d_2 - d_1b_2).
\]
(6.2)

Given the angle \( \theta \) and the two instantaneous curvatures \( \kappa_- \) and \( \kappa_+ \), the coefficients \( a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \) in the parameterization (6.1) can be recovered up to rotation and translation — that is, the coefficients corresponding to a contour which is the image of \( \gamma \) under translation and rotation can be readily derived from \( \theta, \kappa_- \), and \( \kappa_+ \). Note that this depends on our assumption that the parameterization is with respect to arclength (which reduces the number of free variables to 4).

We shall denote by \( \Gamma(\theta, \kappa_-, \kappa_+) \) the space of curves of this type parameterized over the interval \([-1, 1]\) with angle \( \theta \) and instantaneous curvatures \( \kappa_- \) and \( \kappa_+ \). Moreover, we shall associate with every \( \gamma \in \Gamma(\theta, \kappa_-, \kappa_+) \) the kernel defined for \((x, y) \in \gamma \times \gamma\) by
\[
K_\gamma(x, y) \left( \frac{i}{4} \nu(y) \cdot \nabla_y + 1 \right) H_0(k|x - y|),
\]
where \( \nu(y) \) is the outward pointing unit normal derivative of \( \gamma \) at \( y \) and \( H_0 \) is the Hankel function of zeroth order. The following is immediate:
THEOREM 6.1. If \( \gamma_0 \in \Gamma(\theta, \kappa_0, \kappa_+ \) is a curve such that all coefficients in the expansion (6.1) of order higher than 2 are 0 and \( \gamma \in \Gamma(\theta, \kappa_0, \kappa_+ \), then \( K_\gamma - K_{\gamma_0} \) is piecewise smooth.

It follows that a basis for the solutions of the boundary integral equation (4.1) for the set \( \Gamma(\theta_0, \kappa_-, \kappa_+ \) can be obtained by augmenting the charge basis elements with functions of the form

\[
p(x) + q(x)
\]

where \( p(x) \) is a polynomial over a given degree on the interval \([-1, 0]\) and \( q(x) \) is a polynomial over a given degree on the interval \((0, 1]\). Quadratures built on this augmented basis can then accurately integrate any \( \gamma \in \Gamma(\theta, \kappa_0, \kappa_+ \), assuming the degree of the polynomials is chosen to be sufficiently high. In practice, of course, there are limitations imposed on the curves which can be treated by the degree of the polynomials \( p(x) \) and \( q(x) \) chosen, but these limitations are no more severe than those present in any scheme for the discretization of boundary integral equations on smooth curves.

Let \( \Gamma_f \) be the space of corner regions whose oriented angles and instantaneous curvatures lie in given ranges

\[
\theta \in [\theta_m, \theta_M], \\
\kappa_- , \kappa_+ \in [\kappa_m, \kappa_M].
\]

Let \( \{ \theta_1 = \theta_m, \theta_2, \ldots, \theta_n = \theta_M \} \) be samples of \([\theta_m, \theta_M]\), and similarly let \( \{ \kappa_1 = \kappa_m, \kappa_2, \ldots, \kappa_\beta = \kappa_M \} \) be samples of \([\kappa_m, \kappa_M]\). Let \( \Phi_{ijk} \) be the augmented charge basis for the corner region with \( \kappa_- = \kappa_i, \kappa_+ = \kappa_j, \) and \( \theta = \theta_k \). The collection \( \Phi = \bigcup_{i,j,k} \Phi_{ijk} \) then forms an oversampled charge basis that (for sufficiently large \( \alpha \) and \( \beta \)) will be approximately correct for curves in \( \gamma \in \Gamma_f \). This follows from rank deficiency arguments; i.e., the dimension of the basis necessary to span solutions to a given precision \( \epsilon \) is finite even as the parameters \( \theta, \kappa_+ \), and \( \kappa_- \) are allowed to vary over small intervals. By proceeding as in the previous section beginning with the basis \( \Phi \), a collection of quadratures for the discretization of boundary integral equations over such regions — that is, the far, near, and diagonal regimes described in the preceding section — can be obtained.

7. Numerical results

The algorithm of this paper for the construction of universal quadratures for domains with corners was implemented in Fortran 77 as was (the obvious) modification of that algorithm for the construction of purpose-made quadratures for specific curve segments. The resulting code was compiled with the Lahey/Fujitsu Linux64 Fortran Compiler Release 8.10a. Timings were performed on a PC with an Intel Core i7 2.67 GHz processor and 12GB of memory and refer to wall clock time. No attempt was made to parallelize any of the code.

All boundary integral equations were inverted using a simple direct solver which is \( O(n^2) \) in the number of discretization nodes \( n \). The precise algorithm will be reported at a later date.

7.1. Purpose-made quadratures.

7.1.1. A Laplace Neumann problem. In this numerical experiment, we solved the exterior Neumann problem

\[
\Delta u(x) = 0 \text{ for } x \in \Omega_3^c \\
\lim_{x \to p} \frac{\partial u}{\partial \nu}(x) = f(p) \text{ for } p \in \partial \Omega_3,
\]

where \( \Omega_3 \) is the domain shown in Figure 5c, \( \frac{\partial}{\partial \nu} \) denotes differentiation with respect to the outward pointing normal vector, and the boundary data \( f(p) \) was taken to be the outward normal derivative on \( \partial \Omega_3 \) of a potential function \( u \) arising from 5 charges placed randomly in the interior of \( \Omega_3 \). The solution \( u(x) \) of (7.1) was represented in the form

\[
u(x) = \frac{1}{2\pi} \int_{\partial \Omega_3} \log |x - y| \sigma(y) \, dS(y).
\]

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A total of 871 quadratures nodes were used to discretize the resulting boundary integral equation

\[ \frac{1}{2} \sigma(x) + \frac{1}{2\pi} \int_{\partial \Omega_3} \nabla_x \log |x - y| \cdot \nu(x) \sigma(y) \, dS(y) = f(x). \] (7.3)

The 12 corner regions were discretized using purpose-made quadratures, which ranged in size from 17 to 22 nodes, while the smooth portions of the curve were discretized using 630 piecewise Legendre quadrature nodes. Figure 3 shows four charge basis functions obtained in the process of constructing a purpose-made quadrature for one of the corner regions. Table 1 shows that nodes and weights of a Chebyshev quadrature for the basis functions at one corner point of \( \partial \Omega_3 \).

In order to access the accuracy of the discretization, the difference between the true potential function \( v \) and the single layer potential

\[ u(x) = \frac{1}{2\pi} \int_{\partial \Omega_3} \log |x - y| \sigma(y) \, dS(y), \]

where \( \sigma \) is the computed inverse of equation (7.3), was measured at a collection of 100 randomly chosen points in the exterior of the domain \( \Omega_3 \) as well as at 300 points on a circle \( \Gamma \) enclosing the domain \( \Omega_3 \). The largest absolute error was found to be \( 1.78 \times 10^{-13} \) and the estimated relative \( L^2(\Gamma) \) error \( \|u - v\|_2/\|v\|_2 \) was \( 6.24 \times 10^{-13} \). The condition number of the discrete system of equations, as computed using the LAPACK DGESVD routine, was found to be 70.83.
The solution $u(x)$ of the problem (7.4) was represented in the form

$$ u(x) = \int_{\partial \Omega} \left( \frac{i}{4} \nu(y) \cdot \nabla y + 1 \right) H_0(k|x-y|) \sigma(y) \, dS(y), $$

(7.5)

where $\nu(y)$ denotes the outward pointing normal vector to the contour $\partial \Omega$ at the point $y$, $\nabla y$ denotes the gradient in the $y$ variable, and $H_0$ is the Hankel function of the first kind of order 0. This representation leads to the boundary integral equation

$$ -\frac{1}{2} \sigma(x) + \int_{\partial \Omega} \left( \frac{i}{4} \nu(y) \cdot \nabla y + 1 \right) H_0(k|x-y|) \sigma(y) \, dS(y) = f(x). $$

(7.6)

This integral equation was discretized using a combination purpose-made quadratures, one for each of the 3 corner regions of $\partial \Omega$, and 30-point piecewise Legendre quadratures for the smooth portions of the contour. A total of 1163 quadratures nodes were used; the purpose-made quadratures were 70, 68, and 65 points. The error in the representation (7.5) was measured at 100 random points in the exterior of

<table>
<thead>
<tr>
<th>$x_j$</th>
<th>$w_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.138738685113948E - 02</td>
<td>0.4354041406838792E - 02</td>
</tr>
<tr>
<td>+0.138738685113948E - 02</td>
<td>0.435406497743280E - 02</td>
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<td>0.1631770821029729E - 01</td>
</tr>
<tr>
<td>+0.1109910940891159E - 01</td>
<td>0.1631762274899377E - 01</td>
</tr>
<tr>
<td>-0.319514017880308E - 01</td>
<td>0.2061086888028045E - 01</td>
</tr>
<tr>
<td>+0.319514017880308E - 01</td>
<td>0.2061131140466802E - 01</td>
</tr>
<tr>
<td>-0.467134268449031E - 01</td>
<td>0.1620896857036356E - 01</td>
</tr>
<tr>
<td>+0.467134268449031E - 01</td>
<td>0.1620820425641860E - 01</td>
</tr>
<tr>
<td>-0.720467277151267E - 01</td>
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</tr>
<tr>
<td>+0.720467277151267E - 01</td>
<td>0.284665156102715E - 01</td>
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<td>-0.9803504448964495E - 01</td>
<td>0.2605095225788128E - 01</td>
</tr>
<tr>
<td>+0.9803504448964495E - 01</td>
<td>0.2604992187406545E - 01</td>
</tr>
<tr>
<td>-0.1279931230137422E + 00</td>
<td>0.3231830706262308E - 01</td>
</tr>
<tr>
<td>+0.1279931230137422E + 00</td>
<td>0.3231928817893967E - 01</td>
</tr>
<tr>
<td>-0.158014909792693E + 00</td>
<td>0.2560865570860328E - 01</td>
</tr>
<tr>
<td>+0.158014909792693E + 00</td>
<td>0.2560743846558139E - 01</td>
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<td>-0.1755086868810900E + 00</td>
<td>0.1138011381573551E - 01</td>
</tr>
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<td>+0.1755086868810900E + 00</td>
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<tr>
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<td>0.1527170385426392E - 01</td>
</tr>
<tr>
<td>+0.1891329922690670E + 00</td>
<td>0.1527098019054412E - 01</td>
</tr>
<tr>
<td>-0.1995343844090571E + 00</td>
<td>0.3627120387824743E - 02</td>
</tr>
<tr>
<td>+0.1995343844090571E + 00</td>
<td>0.3627296192599455E - 02</td>
</tr>
</tbody>
</table>

Table 1: The 22 nodes and weights of a Chebyshev quadrature for the basis functions computed for one corner region of the boundary of $\partial \Omega$.  

7.1.2. A Helmholtz Dirichlet problem. In this example, we solved the Helmholtz exterior Dirichlet problem

$$ \Delta u(x) + k^2 u(x) = 0 \text{ for } x \in \Omega^c,$$

$$ \lim_{x \to y \in \partial \Omega} u(x) = f(p) \text{ for } x \in \Omega^c,$$

(7.4)

where $\Omega$ is the “shark’s fin” domain shown in Figure 4, the wavenumber $k$ is 20, and $f(p)$ is taken to be the restriction to the boundary $\partial \Omega$ of a acoustic potential $u(x)$ generated by 5 random charges placed in the interior of the domain $\Omega$. The domain $\Omega$ is approximately 80 wavelengths by 80 wavelengths in size.
Figure 4: The “shark’s fin” domain $\Omega$ under consideration in Section 7.1.2.

<table>
<thead>
<tr>
<th>Range of angles (radians)</th>
<th>Charge basis dimension</th>
<th>Far quadrature nodes</th>
<th>Largest diagonal quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.58905 - 0.78540</td>
<td>39</td>
<td>66</td>
<td>70</td>
</tr>
<tr>
<td>0.78540 - 0.98175</td>
<td>36</td>
<td>56</td>
<td>60</td>
</tr>
<tr>
<td>1.1781 - 1.3744</td>
<td>33</td>
<td>56</td>
<td>62</td>
</tr>
<tr>
<td>1.4137 - 1.5708</td>
<td>31</td>
<td>56</td>
<td>60</td>
</tr>
<tr>
<td>1.5708 - 1.7279</td>
<td>30</td>
<td>54</td>
<td>58</td>
</tr>
<tr>
<td>2.3562 - 2.6180</td>
<td>28</td>
<td>52</td>
<td>55</td>
</tr>
<tr>
<td>4.7124 - 4.8645</td>
<td>31</td>
<td>56</td>
<td>60</td>
</tr>
</tbody>
</table>

Table 2: Charge basis and quadrature sizes for seven “universal” quadratures for the exterior Laplace Dirichlet problem.

The domain $\Omega$ as well as at 300 points on a circle $\Gamma$ enclosing $\Omega$. The largest absolute error was found to be $3.56 \times 10^{-13}$ while the estimated relative $L^2(\Gamma)$ error was found to be $7.67 \times 10^{-12}$. The condition number of the discrete system, as computed by the LAPACK routine ZGESVD, was 918.54.

7.2. Universal quadratures. Using the procedures described in Sections 5 and 6, universal quadratures were constructed for the discretization over domains with corners of the boundary integral equations

$$-\frac{1}{2} \sigma(x) + \frac{1}{2\pi} \int_{\partial \Omega} (\nabla_y \log |x - y| \cdot \nu(y) + 1) \sigma(y) \, dS(y) = u(x),$$

$$\frac{1}{2} \sigma(x) + \frac{1}{2\pi} \int_{\partial \Omega} \nabla_x \log |x - y| \cdot \nu(y) \sigma(y) \, dS(y) = u(x),$$

and

$$-\frac{1}{2} \sigma(x) + \int_{\partial \Omega} \left( \frac{i}{4} \nu(y) \cdot \nabla_y + 1 \right) H_0(k|x - y|) \sigma(y) \, dS(y) = u(x),$$

where $H_0$ is the Hankel function of zeroth order and $\nu(y)$ denotes the outward pointing unit normal vector of $\partial \Omega$ at the point $y$. Several collections of quadratures were constructed for the boundary integral equation (7.9), one for each of several different ranges of the wavenumbers $k$ in the interval $[1, 20]$. The integral equation (7.7) arises from the solution of the exterior Dirichlet problem for Laplace’s equation, (7.8) arises from the solution of the exterior Neumann problem for Laplace’s equation, and (7.9) is associated with the exterior Dirichlet problem for the Helmholtz equation with wavenumber $k$ (see [9], for instance).

In the case of the exterior Dirichlet problem for Laplace’s equation, we constructed 22 quadrature formulae for various ranges of angles covering the interval $[1, 2\pi - .1]$. In each case, the range of curvatures was taken to be $[-1, 1]$, which was sufficient for the examples given here. Five angles and four curvatures were sampled in order to construct the charge bases used to compute each formula. Table 2 gives details for 7 of the 22 quadratures constructed for the Laplace exterior Dirichlet problem.
Figure 5: The domains under consideration in Section 7.2. The Pacman domain is approximately 4 units by 4 units in size, while the dimensions of the inkblot domain are approximately 6 by 6. The tank domain is approximately 6 units in length by 3 units in height.

<table>
<thead>
<tr>
<th>Problem</th>
<th>k</th>
<th>N</th>
<th>T</th>
<th>$E_{pot}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace Dirichlet</td>
<td>-</td>
<td>284</td>
<td>0.02</td>
<td>$4.41 \times 10^{-13}$</td>
<td>32.7</td>
</tr>
<tr>
<td>Laplace Neumann</td>
<td>-</td>
<td>514</td>
<td>0.22</td>
<td>$5.42 \times 10^{-14}$</td>
<td>1.03 $\times 10^{6}$</td>
</tr>
<tr>
<td>Helmholtz Dirichlet</td>
<td>0.1</td>
<td>364</td>
<td>0.71</td>
<td>$1.11 \times 10^{-13}$</td>
<td>634</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>370</td>
<td>0.74</td>
<td>$6.90 \times 10^{-14}$</td>
<td>346</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>375</td>
<td>0.79</td>
<td>$7.75 \times 10^{-14}$</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>380</td>
<td>0.83</td>
<td>$1.38 \times 10^{-13}$</td>
<td>500</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>570</td>
<td>1.35</td>
<td>$5.57 \times 10^{-12}$</td>
<td>39.1</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>572</td>
<td>1.49</td>
<td>$7.98 \times 10^{-14}$</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>939</td>
<td>2.59</td>
<td>$7.81 \times 10^{-14}$</td>
<td>244</td>
</tr>
</tbody>
</table>

Table 3: Computational results obtained for the domain $\Omega_1$

For each of the domains shown in Figure 5, the exterior Dirichlet and Neumann problems for Laplace’s equation and the exterior Dirichlet problem for the Helmholtz equation at various wavenumbers were solved. In each case, the boundary data was taken to be an electromagnetic or acoustic potential generated by a collection of 5 random charges in the interior of the domain and the boundary integral equation was discretized using universal quadratures for corner regions and piecewise Legendre quadratures on the smooth portions of the curve. The resulting solution was tested by comparing the value of the true potential function to the computed solution at 100 randomly chosen points in the exterior.
of the domain. Tables 3, 4, and 5 present the results. The following quantities are reported for each problem:

- k the wavenumber for the problem (if applicable);
- N is the number of discretization nodes required for the problem;
- T is the wall clock time required to solve the integral equation;
- $E_{\text{pot}}$ the largest absolute error measured in the computed potential; and
- $\kappa$ is the condition number of the discrete system of equations, as computed by either LAPACK’s DGESVD or ZGESVD routine.

Remark 7.1. The ill-conditioning of the systems arising from Neumann problems for Laplace’s equation is a subject of ongoing investigation by the authors.

7.3. A final example. As a final example, we used universal quadratures to solve an exterior Dirichlet problem for the Helmholtz equation at wavenumber $k = 1$ on the domain $\Omega_3$ shown in Figure 6. The boundary of $\Omega_3$ is formed from two Lipschitz functions and has 40 corner points. The domain $\Omega_3$ is approximately 3 wavelengths in width and 3 wavelengths in height. The boundary data for the problem was taken to be an acoustic potential generated by 5 charges randomly placed in the interior of the domain.

The discretization of the boundary integral equation (7.9), which was used to solve the boundary value problem, required 7794 discretization nodes. Between 54 and 70 discretization nodes were required

<table>
<thead>
<tr>
<th>Problem</th>
<th>k</th>
<th>N</th>
<th>T</th>
<th>$E_{\text{pot}}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace Dirichlet</td>
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<td>1256</td>
<td>0.25</td>
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<tr>
<td>Laplace Neumann</td>
<td>-</td>
<td>1444</td>
<td>1.78</td>
<td>4.60×10^{-12}</td>
<td>5.68×10^7</td>
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<tr>
<td>Helmholtz Dirichlet</td>
<td>0.1</td>
<td>1456</td>
<td>4.71</td>
<td>1.72×10^{-12}</td>
<td>722</td>
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<tr>
<td>Helmholtz Dirichlet</td>
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<td>1424</td>
<td>4.71</td>
<td>1.72×10^{-12}</td>
<td>711</td>
</tr>
<tr>
<td>Helmholtz Dirichlet</td>
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<td>1528</td>
<td>5.12</td>
<td>3.55×10^{-12}</td>
<td>461</td>
</tr>
<tr>
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<td>5.19</td>
<td>7.87×10^{-12}</td>
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</tr>
<tr>
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<td>2504</td>
<td>10.76</td>
<td>6.24×10^{-14}</td>
<td>696</td>
</tr>
<tr>
<td>Helmholtz Neumann</td>
<td>5</td>
<td>2464</td>
<td>10.57</td>
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<td>2476</td>
<td>11.27</td>
<td>5.88×10^{-12}</td>
<td>43.7</td>
</tr>
<tr>
<td>Helmholtz Neumann</td>
<td>20</td>
<td>4492</td>
<td>31.23</td>
<td>6.75×10^{-12}</td>
<td>263</td>
</tr>
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</table>

Table 4: Computational results obtained for the domain $\Omega_2$

<table>
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<tr>
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<th>N</th>
<th>T</th>
<th>$E_{\text{pot}}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1630</td>
<td>0.76</td>
<td>4.08×10^{-12}</td>
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</tr>
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<td>Laplace Neumann</td>
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<td>1781</td>
<td>3.95</td>
<td>4.10×10^{-14}</td>
<td>7267</td>
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<tr>
<td>Helmholtz Dirichlet</td>
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<td>8.63</td>
<td>2.67×10^{-13}</td>
<td>1379</td>
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<tr>
<td>Helmholtz Dirichlet</td>
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<td>1385</td>
<td>9.29</td>
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<td>573</td>
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<td>11.83</td>
<td>4.92×10^{-13}</td>
<td>194</td>
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<td>2746</td>
<td>25.49</td>
<td>1.13×10^{-13}</td>
<td>727</td>
</tr>
<tr>
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<td>2678</td>
<td>24.14</td>
<td>8.79×10^{-13}</td>
<td>149</td>
</tr>
<tr>
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<td>3132</td>
<td>28.12</td>
<td>7.60×10^{-14}</td>
<td>226</td>
</tr>
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</table>

Table 5: Computational results obtained for the domain $\Omega_3$
Figure 6: The domain $\Omega_4$, formed from the graph of two Lipschitz functions.

per corner. Inversion of the resulting discrete linear system required 71.83 seconds and resulting in an approximate charge distributions which, when compared with the true potential function at a collection of 100 randomly placed points in the exterior of the domain, yielded a maximum absolute error of $2.13 \times 10^{-12}$. The condition number of the resulting discrete system is 748.

8. Conclusions

We have introduced an algorithm for the construction of “universal quadratures” for the efficient discretization of certain Laplace and Helmholtz boundary integral equations over a very general class of domains with corner points. Once they have been constructed, these quadratures reduce the complexity of solving a boundary integral equation on such domains to approximately that of solving the same equation over well-behaved smooth domains. While the implementation presented here is for certain Laplace and Helmholtz boundary value problems over planar domains with corners and assumes a specific form for the right-hand sides, the extension of our approach to other boundary value problems, other pathological domains, and to other classes of right-hand sides is straightforward and will be reported at a later date. Moreover, certain aspects of our scheme — specifically the basis construction procedure of Section 4 — generalize to the case of surfaces with singularities.

9. Acknowledgments

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10. Appendix: An elementary rank of interaction estimate

In this short appendix, we will denote by $T$ the integral operator taking functions on $[-1, 0]$ to functions on $[0, 1]$ defined by

$$Tf(x) = \int_{-1}^{0} \log |x - y| f(y) dy.$$ 

The kernel $K(x, y) = \log |x - y|$ is in $L^2([0, 1] \times [-1, 0])$; in fact,

$$\int_{0}^{1} \int_{-1}^{0} \log^2 |x - y| dy \, dx = \frac{7}{2} + 2 \log^2(2) - \log(64).$$

It follows that $T$ is a Hilbert-Schmidt integral operator and is therefore compact as an operator $L^2([-1, 0]) \rightarrow L^2([0, 1])$. Since compact operators admit singular value decomposition, we can now make the following definition:

**Definition 10.1.** We call the greatest positive integer $m$ such that the $m$ largest singular values of $T$ are greater than or equal to $\epsilon$ the $\epsilon$-rank of $T$.

In what follows, we shall use repeatedly the fact that if a rank $n$ operator $S_n$ can be found such that $\|S - S_n\|_2 < \epsilon$, then the $(n + 1)$-st singular value of $S$ is bounded above by $\epsilon$ and the $\epsilon$-rank of $S$ is, in this event, bounded above by $n$. This follows from the optimality of rank $n$ approximation of an operator via singular value decomposition.

The purpose of this section is the proof of the following statement on the decay of the singular values of $T$.

**Theorem 10.1.** The $\epsilon$-rank of the operator $T$ behaves as $O\left(\log(\epsilon)^2\right)$ as $\epsilon \rightarrow 0$.

We shall prove Theorem 10.1 with a sequence of lemmas, starting with the following statement on the approximation of $T$ via a truncated integral operator.

**Lemma 10.1.** If $0 < \alpha < 1$ and $T_{\alpha} : L^2([-1, 0]) \rightarrow L^2([0, 1])$ is defined by

$$T_{\alpha}f(x) = \int_{-1}^{-\alpha} \log |x - y| f(y) dy$$

then $\|T - T_{\alpha}\|_2 \leq \sqrt{2\alpha}$

**Proof.** For $x > 0$ we have:

$$|Tf(x) - T_{\alpha}f(x)|^2 = \left| \int_{-\alpha}^{0} \log |x - y| f(y) dy \right|^2 \leq \|f\|_2^2 \int_{-\alpha}^{0} \log^2 (x - y) dy = \|f\|_2^2 g(x, \alpha),$$

where $g(x, \alpha) = 2\alpha + 2x \log(x) - x \log^2(x) + (\alpha + x)(-2 + \log(\alpha + x)) \log(\alpha + x)$ and $\|f\|_2$ denotes the $L^2([-1, 0])$ norm of $f$. Now set

$$G(\alpha) = \int_{0}^{1} g(x, \alpha) = \frac{1}{2} (7\alpha + 3\alpha^2 \log(\alpha) - \alpha^2 \log(\alpha) + (1 + \alpha)^2 (-3 + \log(1 + \alpha)) \log(1 + \alpha)).$$

By examining the Taylor series of $G(\alpha)$ about $\alpha$, we see that for $0 < \alpha < 1$,

$$G(\alpha) \leq 2\alpha.$$
LEMMA 10.2. If $I$ is a compact interval contained in $[-1, 1]$ and $J$ is a compact interval contained in $[3, \infty)$, then the $\epsilon$-rank of the operator $S : L^2(I) \to L^2(J)$ defined by

$$Sf(x) = \int_I \log |x - y|f(y)dy$$

is bounded above by $[-\log_3(\epsilon)] + 1$.

**Proof.** Expanding $\log |x - y|$ in a Taylor series about $y = 0$ gives

$$Sf(x) = \log(x) \int_I f(y)dy - \sum_{k=1}^{\infty} \frac{x^{-k}}{k} \int_I y^k f(y)dy.$$ 

For integers $N > 1$ define $S_N$ by

$$S_N f(x) = \log(x) \int_I f(y)dy - \sum_{k=1}^{N-1} \frac{x^{-k}}{k} \int_I y^k f(y)dy.$$ 

Then $S_N$ is of rank $N$ since its range is spanned by $\log(x)$ and $x^{-k}$, $k = 1, \ldots, N - 1$. Moreover, for $x > 3$ we have

$$|Sf(x) - S_N f(x)|^2 \leq \left( \sum_{k=N}^{\infty} \frac{x^{-k}}{k} \left| \int_I y^k f(y)dy \right| \right)^2 \leq \|f\|_2^2 \left( \sum_{k=N}^{\infty} \frac{x^{-k}}{k} \int_{-1}^{1} y^{2k}dy \right)^2 = \|f\|_2^2 \left( \sum_{k=N}^{\infty} \frac{2 \cdot x^{-k}}{k(2k+1)} \right)^2 \leq \|f\|_2^2 \left( \frac{x^{1-N}}{x-1} \right)^2,$$

where $\|f\|_2$ denotes the $L^2([-1, 1])$ norm of the function $f$.

It follows that

$$\|Sf(x) - S_N f(x)\|_2^2 \leq \|f\|_2^2 \int_J \frac{x^{2-2N}}{(x-1)^2}dx \leq \frac{\|f\|_2^2}{4} \int_J x^{2-2N}dx = 3^{1-N} \|f\|_2^2 4(2N-3) \leq \frac{9}{4} \|f\|_2^2 3^{-2N},$$

and so

$$\|S - S_N\|_2 \leq \frac{3^{1-N}}{2}. \quad (10.1)$$

If $N$ is chosen so that $N \geq -\log_3(\epsilon) + 1$, then the error in (10.1) guaranteed to be less than $\epsilon$, which implies that the $(N + 1)$-st singular value of $S$ is less than $\epsilon$. □

The following can be obtained via an obvious modification of the preceding result:

LEMMA 10.3. If $\phi$ is an affine mapping $\mathbb{R} \to \mathbb{R}$, $I$ is a compact interval contained in $[-1, 1]$ and $J$ is a compact interval contained in $[3, \infty)$, then $\epsilon$-rank of the operator $S : L^2(\phi(I)) \to L^2(\phi(J))$ defined by

$$Sf(x) = \int_{\phi(I)} \log |x - y|f(y)dy$$

is bounded above by $[-\log_3(\epsilon)] + 1$. The proof follows in a similar manner as above.
is bounded above by $[-\log_3(\epsilon)] + 1$.

**Lemma 10.4.** The $\epsilon$-rank of the operator $T_\alpha$ defined by

$$T_\alpha f(x) = \int_{-1}^{-\alpha} \log|x-y|f(y)dy,$$

where $0 < \alpha < 1$, is bounded above by $[\log_2(\alpha)] [\log_3(\epsilon) - 1]$.

**Proof.** The integral operator $T_\alpha$ can be decomposed as

$$T_\alpha f(x) = \sum_{k=0}^{N} \int_{I_k} \log|x-y|f(y)dy,$$

where $N = -\log_2(\alpha)$ and $I_k$ is defined for $k = 0, \ldots, N$ by

$$I_k = [-2^{-k}, \min(-\alpha, -2^{-(k+1)})].$$

The claim now follows from the (obvious) fact that the rank of the sum of two operators is bounded above by the sum of ranks of the operators. □

We are now in a position to prove Theorem 10.1. If we choose $\alpha < \epsilon^2/8$, then $\|T - T_\alpha\|_2 < \epsilon/2$ by Lemma 10.1. Moreover, the $2^{-\epsilon}$-rank of $T_\alpha$ is bounded above by

$$N = \log_2 \left( \frac{\epsilon^2}{8} \right) \left[ \log_3 \left( \frac{\epsilon}{2} \right) - 1 \right] = O \left( \log(\epsilon)^2 \right)$$

by Lemma 10.4; that is, there exists a rank-$N$ operator $\tilde{T}_\alpha$ such that $\|T_\alpha - \tilde{T}_\alpha\|_2 < \epsilon/2$. It follows that the $\epsilon$-rank of $T$ is bounded above by $N$ since we have

$$\|T - \tilde{T}_\alpha\|_2 \leq \|T - T_\alpha\|_2 + \|T_\alpha - \tilde{T}_\alpha\|_2 < \epsilon.$$

**References**


