Cluster Algebras

Some Context: Total Positivity

April 12th 2021
**Total Positivity** 1  
Original Motivation for Cluster Algebras.

**Def:** A real valued \( n \times n \) matrix is **totally positive** (non-negative) if all of its minors are positive (non-negative).

- Show up in classical mechanics, probability, asymptotic representation theory, combinatorics

- Totally positive (non-negative) matrices form a semigroup in \( SL_n \mathbb{R} \).
Total Positivity 2

• Generalize to varieties: Take a complex variety $X$, and a family $\Delta$ of ‘important’ regular functions $X \to \mathbb{C}$.

$$X_{\geq 0} := \left\{ x \in X \mid \Delta(x) \geq 0 \text{ for all } \Delta \in \Delta \right\}$$

• $X = \text{Mat}_n \mathbb{C}$, $\Delta = \{ \text{all minors} \}$ gives the original notion.
Example 1

2x2 matrices

A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}

Minors:

relation: \( ad = \Delta + bc \) means we only need to check
Example 2

\[ N < SL_3 \mathbb{C} \text{ unipotent upper triangular} \]

\[ N_{x_0} = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} e SL_3 \mathbb{C} \right\} \]

Relation: \( xz = (xz-y) + y \)
Grassmannians $\text{Gr}_{k,m}(\mathbb{C})$ 1

Def: The Grassmannian $\text{Gr}_{k,m}(\mathbb{C})$ is the space of all $k$-dimensional subspaces of an $m$-dimensional complex vector space $V$.

Ex: $\text{Gr}_{1,2}(\mathbb{R})$: all lines through the origin in $\mathbb{R}^2$
Grassmannians $\text{Gr}_{k,m}(\mathbb{C})$ 2

Appropriate notion of totally positive / non-negative points.

The concept of **row span** lets us work with matrices:

1. $V \subseteq \mathbb{C}^m$ via a choice of basis.
   
2. Let $z = [\cdots]$ be a $k \times m$ matrix of rank $k$.

   Its rows span a $k$-dim'l subspace of $\mathbb{C}^m$.

   So it identifies a point $[z] \in \text{Gr}_{k,m}(\mathbb{C})$

   Example: $[1 \ 2]$
**Plücker Coordinates 1**

Def: Given a $k$-element subset $J \in \{1, \ldots, m\}$, the **Plücker coordinate** $P_J(z)$ is the determinate of the $k \times k$ submatrix of $z$ determined by columns $J$.

$$z = \begin{bmatrix} \vdots & \vdots & \vdots \\ \end{bmatrix} \quad \sim \quad P_{\{1,4,5\}}(z) = \text{det}(\begin{bmatrix} \vdots \vdots \vdots \end{bmatrix})$$
Plücker Coordinates 2

Note: The collection \( \Delta := \{ P_J(z) \mid |J| = k \} \) depends only on the row span of \( z \) (up to a common rescaling.)

Def: The totally positive Grassmannian \( \text{Gr}^+_k,m \subset \text{Gr}_k,m \) consists of those points whose Plücker Coordinates can be chosen to all be positive real numbers.

\( \star \) i.e. they all have the same sign.

- These are \( \binom{m}{k} \) plücker coordinates to check for each matrix! Can this number be reduced like in the examples?
**Optimal Tests for Total Positivity**

Focus on $\text{Gr}_{2,m \mathbb{C}}$.

Key feature: There are relations between the Plücker coordinates.

Three-term Grassmann-Plücker relations:

$$P_{i} P_{j} P_{k} P_{l} = P_{i} P_{j} P_{k} + P_{i} P_{j} P_{k}$$

$$1 \leq i < j < k < l \leq m$$

$z \left[ \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ i & j & k & l \end{array} \right]$

- We can use this to find optimal tests for total positivity.
Plücker Coordinates + Triangulations

Consider convex \( m \)-gon, vertices labeled clockwise.

- Chord connecting \( i \) and \( j \) ↔ Plücker coordinate \( p_{ij} \).
- A triangulation \( T \) of the \( m \)-gon has \( m-3 \) diagonals. Together with the \( m \) sides, it produces \( 2m-3 \) Plücker coordinates \( \tilde{x}(T) \) "extended cluster".
Example and Notation

\[ \tilde{\chi}(\tau) = \{ p_{12}, p_{23}, \ldots, p_{78}, p_{81} \} \]
Extended Clusters and Positivity

Thm: Each Plücker coordinate $P_{ij}$ can be written as a subtraction free rational expression in the elements of a given extended cluster $\tilde{X}(T)$.

Hence, if the $2m-3$ Plücker coordinates in $\tilde{X}(T)$ are positive for a $2\times m$ matrix $z$, then all of its Plücker coordinates are positive.

In other words: each $\tilde{X}(T)$ can detect total positivity.
Proof: Flips and Exchanges

(1) Each $P_{ij}$ shows up in $\hat{x}(T)$ for some triangulation $T$

(2) Any two triangulations are related by a series of flips:

(3) A flip exchanging diagonals $ik$ and $jl$ exchanges $P_{ik}$ and $P_{jl}$ in the extended clusters. But:

$$P_{ik} P_{jl} = P_{ij} P_{kl} + P_{il} P_{jk} \rightarrow P_{ik} = \frac{P_{ij} P_{kl} + P_{il} P_{jk}}{P_{jl}}$$

Subtraction free! \qed
Concluding Remarks:

- Clusters arose from the quest for efficient tests of total positivity

Key features:

- Distinguished generators (minors, Plücker coordinates)
- Grouping into overlapping extended clusters
- Combinatorial description of extended clusters
- Exchange relations between extended clusters - subtraction free.
Thank you!