The AKLT Model

Lecture 5

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This talk will follow pg. 26-29 of *Lieb-Robinson Bounds in Quantum Many-Body Physics* by B. Nachtergaele and R. Sims

Introduction

The AKLT model, introducd by Affleck, Kennedy, Lieb, and Tasaki in 1987, is a spin-1 chain with Hamiltonian

$$H_{[a,b]}^{AKLT} = \sum_{x=a}^{b-1} [\frac{1}{3} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2]$$

This was the first model to satisfy the Haldane phase characteristics.

Spin

The spin of a particle describes its possible angular momentum values. The spin value of a particle is a half-integer $j = \frac{n}{2}$. If a particle has spin j its possible values of angular momentum are

$$j, j-1, j-2, \ldots, -j$$

- The irreducible Lie-algebra representation of $\mathfrak{su}_2$, which describes a particle of spin j, has dimension $d = 2j+1$.
- The three generators for the Lie-algebra representation are denoted S^1, S^2, S^3, and are often combined in the vector $\mathbf{S} = (S^1, S^2, S^3)$.
- A particle of spin j has a local Hilbert space $\mathcal{H}_x = \mathbb{C}^d$.

Spin-1 Example

In the $j = 1$ case, the local Hilbert space is $\mathbb{C}^3$, and the spin matrices that generate the irreducible Lie algebra are:

$$S^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Casimir Operator

The Casimir Operator for a representation of $\mathfrak{su}_2$ is the element $\mathbf{S}^2 = (S^1)^2 + (S^2)^2 + (S^3)^2$. Using the relation $[S^i, S^j] = i\varepsilon_{ijk}S^k$, we can show that $[\mathbf{S}^2, S^i] = 0$ for $i = 1, 2, 3$. Recall, for any $u(g) \in \mathfrak{su}_2$, we can write $u(g) = e^{i(\theta_1 S^1 + \theta_2 S^2 + \theta_3 S^3)}$. Expanding $u(g)$ into its Taylor series we can deduce that $[\mathbf{S}^2, u(g)] = 0$.

In the case of an irreducible representation of $\mathfrak{su}_2$, Schur's Lemma indicates that $\mathbf{S}^2 = c\mathbb{1}$ for some constant c. For the irrep of dimension $2j + 1$ the constant $c = j(j + 1)$.

In the case of a reducible representation, $\mathbf{S}^2 = \sum_j j(j + 1)P^{(j)}$, where each j corresponds to an irrep in the decomposition of the representation, and $P^{(j)}$ is the projection onto that subspace.

Modeling a Quantum Spin Chain

Each individual particle in a quantum spin chain is modeled by an irreducible representation of $\mathfrak{su}_2$. To model the entire chain, we tensor neighboring representations together.

Ex: Let $D^{(j)}$ be the $(2j+1)$-irreducible representation of $\mathfrak{su}_2$ describing a particle of spin j. If we wish to make a quantum spin chain of three consecutive particles with respective spins j_1, j_2, j_3, the $\mathfrak{su}_2$ representation of the chain is

$$D^{(j_1)} \otimes D^{(j_2)} \otimes D^{(j_3)}$$

Modeling a Quantum Spin Chain

We can decompose a tensor of two spin representations into irreducibles using the Clebsch-Gordan series

$$D^{(j_1)} \otimes D^{(j_2)} \cong D^{|j_1-j_2|} \oplus D^{|j_1-j_2|+1} \oplus \ldots \oplus D^{(j_1+j_2)}$$

Ex: The decomposition of two spin-1's is

$$D^{(1)} \otimes D^{(1)} \cong D^{(0)} \oplus D^{(1)} \oplus D^{(2)}$$

The AKLT Model

The AKLT model is a spin-1 chain with isotropic Hamiltonian:

$$H^{AKLT}_{[a,b]} = \sum_{x=a}^{b-1} \left[\frac{1}{3} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right]$$

Each term $h_{x,x+1} = \frac{1}{3} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$ describes a nearest-neighbor interaction.

The local Hilbert space on each site is $\mathbb{C}^3$, so the global Hilbert space is $\mathcal{H}_{[a,b]} = (\mathbb{C}^3)^{\otimes b-a+1}$.

The AKLT Model

Using the Casimir Operator, we compute that

$$\frac{1}{3} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 = P^{(2)}_{x,x+1}$$

Here, $P^{(2)}_{x,x+1}$ is the orthogonal projection onto the spin-2 subspace of the product of two spin-1's located at the sites x, $x+1$.

Since projections are always positive, it follows that the Hamiltonian $H^{AKLT}_{[a,b]} = \sum_{x=a}^{b-1} P^{(2)}_{x,x+1}$ is positive as well. So if we find a state ψ such that $H^{AKLT}_{[a,b]}\psi = 0$, we know this is a ground state. As we shall see such states exist.

The Haldane Phase

Three properties characterize the Haldane phase:

1. A unique ground state in the infinite chain. For the AKLT model, this means that for all choices $\psi_L \in \ker H_{[-L,L]}$, such that $\|\psi_L\| = 1$, and for all finite subchains X and $A \in \mathcal{A}_X$, there is a unique limit $\omega(A)$, where

$$\omega(A) = \lim_{L\to\infty} \langle \psi_L, A\psi_L \rangle$$

The Haldane Phase

2. A finite correlation length $\xi > 0$. For the AKLT model this means there is $C > 0$ such that for any finite subchains X, Y, and all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$,

$$|\omega(AB) - \omega(A)\omega(B)| \leq C\|A\|\|B\|e^{\frac{-d(X,Y)}{\xi}}$$

For the AKLT Model, $\xi = \frac{1}{\ln(3)}$

The Haldane Phase

3. A nonzero spectral gap. For AKLT this means there is a positive number γ such that the distance between the ground state eigenvalue λ_0, and any positive eigenvalue λ, is bounded from below. Thus

$$|\lambda - \lambda_0| \geq \gamma$$

We will discuss why the AKLT model satisfies the first two conditions of the Haldane phase.

Unique Ground State/Finite Correlation Length

Definition: An intertwiner $V : \mathbb{C}^{2s+1} \to \mathbb{C}^{2j+1} \otimes \mathbb{C}^{2s+1}$ is an isometry with the added requirement that, for any $U_g^{(s)} \in D^{(s)}$, we have $VU_g^{(s)} = U_g^{(j)} \otimes U_g^{(s)} V$.

Unique Ground State/Finite Correlation Length

Recall that $D^{(1/2)} \otimes D^{(1/2)} \cong D^{(1)} \oplus D^{(0)}$. So one way to view a spin-1 particle is as the highest irreducible representation of a two spin-1/2 system. We call $s = 1/2$ an auxiliary spin. Let $W : \mathbb{C}^3 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ be the intertwiner embedding $D^{(1)} \subset D^{(1/2)} \otimes D^{(1/2)}$. The embedding W induces an embedding of $A \in M_3$ into $A' \in M_2 \otimes M_2$, where $A' = WAW^*$. Let $\phi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ be the singlet state

$$\phi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Unique Ground State/Finite Correlation Length

For every $n \geq 1$, and any $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$, we define the following vector $\psi_{\alpha\beta}^{(n)} \in \mathcal{H}_{[1,n]}$ by

$$\psi_{\alpha\beta}^{(n)} = (W^* \otimes W^* \otimes \ldots \otimes W^*)(|\alpha\rangle \otimes \phi \otimes \ldots \otimes \phi \otimes |\beta\rangle)$$

Since $W^* : \mathbb{C}^3 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ is an intertwiner, we can show that $(W^*)^{\otimes n} : (\mathbb{C}^3)^{\otimes n} \to (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes n}$ is also an intertwiner.

Unique Ground State/Finite Correlation Length

Notice that $H^{AKLT}_{[1,n]}\psi^{(n)}_{\alpha\beta} = 0$ iff $P^{(2)}_{x,x+1}\psi^{(n)}_{\alpha\beta} = 0$ for all x. Since $P^{(2)}_{x,x+1}$ is the projection onto the spin-2 subspace, if we can show $\psi^{(n)}_{\alpha\beta}$ belongs locally to the spin-1 or spin-0 irrep, this condition will be satisfied. If $\mathbf{S}^2$ is the Casimir operator for $(\mathbb{C}^3)^{\otimes n}$, and $\mathbf{J}^2$ is the Casimir operator for $(\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes n}$, then by properties of intertwiners,

$$\mathbf{S}^2\psi^{(n)}_{\alpha\beta} = (W^*)^{\otimes n}\mathbf{J}^2(|\alpha\rangle \otimes \phi \otimes \ldots \otimes \phi \otimes |\beta\rangle)$$

Since $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$ are the only two components that can contribute a nonzero spin, and $\mathbb{C}^2$ is the Hilbert space of $D^{(1/2)}$, it follows that $\mathbf{S}^2\psi^{(n)}_{\alpha\beta} = c\psi^{(n)}_{\alpha\beta}$, where $c \leq 2$. Therefore, the total spin of $\psi^{(n)}_{\alpha\beta}$ cannot exceed 1 and it follows directly that $P^{(2)}_{x,x+1}\psi^{(n)}_{\alpha\beta} = 0$ for all x.

Unique Ground State/Finite Correlation Length

Recall that $H_{[1,n]}^{AKLT} = \sum_{x=1}^{n-1} P_{x,x+1}^{(2)}$. Since $P_{x,x+1}^{(2)}$ is the projection onto the spin-2 subspace of any two neighboring spin-1 particles, it follows that $P_{x,x+1}^{(2)}\psi_{\alpha\beta}^{(n)} = 0$ for all x. Thus, $\psi_{\alpha\beta}^{(n)}$ is a ground state for $H_{[1,n]}^{AKLT}$.

Exercise: Show that all vectors of the form $\psi_{\alpha\beta}^{(n)}$ span $\ker H_{[1,n]}$. To do this, first consider a chain of length 3, and then induct.

Notice that this result implies that $\dim(\ker H_{[1,n]}) = 4$.

Unique Ground State/Finite Correlation Length

To show uniqueness in the thermodynamic limit, we choose any ground state $\psi^{(n)}_{\alpha\beta}$ and evaluate an arbitrary observable $A_1 \otimes A_2 \otimes \ldots A_n$ on ω_n. Hence

$$\omega_n(A_1 \otimes A_2 \otimes \ldots A_n) = \frac{\langle \psi^{(n)}_{\alpha\beta}, A_1 \otimes A_2 \otimes \ldots A_n \psi^{(n)}_{\alpha\beta} \rangle}{\langle \psi^{(n)}_{\alpha\beta}, \psi^{(n)}_{\alpha\beta} \rangle}$$

<u>Exercise</u>: Show that for any two vectors $|\psi\rangle$, $|\phi\rangle$, $\langle \psi, \phi \rangle = \mathrm{Tr}(|\phi\rangle\langle\psi|)$

Unique Ground State/Finite Correlation Length

We can use the previous exercise to rewite $\omega_n(A_1 \otimes A_2 \otimes \dots A_n)$ as

$$\omega_n(A_1 \otimes A_2 \otimes \dots A_n) = c_n \mathrm{Tr} P_\alpha \mathbb{E}_{A_1} \circ \mathbb{E}_{A_2} \circ \dots \circ \mathbb{E}_{A_n}(P_\beta)$$

The operator $\mathbb{E}_A : M_2 \to M_2$ is defined as $\mathbb{E}_A(B) = V^*(A \otimes B)V$, where $V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$ is the intertwiner defined by

$$V|\alpha\rangle = (W^* \otimes \mathbb{1})(|\alpha\rangle \otimes \phi)$$

Here, c_n is a normalization constant and P_α, P_β are 2×2 matrices depending only on α, β, respectively. It is straightforward that $\mathbb{E}_{\mathbb{1}_3}(\mathbb{1}_2) = \mathbb{1}_2$.

Unique Ground State/Finite Correlation Length

When taking the thermodynamic limit, it is desirable to calculate $\mathbb{E}^k_{\mathbb{1}_3}$ for larges values of k. Hence it is convenient to diagonalize the operator $\mathbb{E}$. Notice that for all 2×2 matrices B we can write B in the basis

$$B = b_0 \mathbb{1}_2 + b_1 \sigma^1 + b_2 \sigma^2 + b_3 \sigma^3$$

Notice that since the Pauli matrices are traceless, it is straightforward to calculate $b_0 = \frac{\mathrm{Tr}(B)}{2}$. To diagonalize $\mathbb{E}$, it is only necessary to know how it acts on σ^i, and we can calculate $\mathbb{E}_{\mathbb{1}}(\sigma^i) = -\frac{1}{3}\sigma^i$.

Unique Ground State/Finite Correlation Length

Using the previous calculations we find that for any $B \in M_2$ $\mathbb{E}_{\mathbb{1}_3}(B)$ diagonalizes as:

$$\mathbb{E}_{\mathbb{1}_3}(B) = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}_2 - \frac{1}{3}(B - \frac{1}{2}\mathrm{Tr}B\mathbb{1}_2)$$

We use this diagonalization to show that

$$\mathbb{E}^k_{\mathbb{1}_3}(B) = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}_2 + \left(\frac{-1}{3}\right)^k (B - \frac{1}{2}\mathrm{Tr}B\mathbb{1}_2)$$

Unique Ground State/Finite Correlation Length

The formula for $\mathbb{E}^k_{\mathbb{1}_3}(B)$ makes it straightforward to calculate the thermodynamic limit

$$\lim_{l\to\infty,\, r\to\infty} \omega_{l+n+r}(\mathbb{1}\otimes\ldots\otimes\mathbb{1}\otimes A_1\otimes\ldots\otimes A_n\otimes\mathbb{1}\otimes\ldots\otimes\mathbb{1})$$
$$=\frac{1}{2}\mathrm{Tr}\mathbb{E}_{A_1}\circ\ldots\circ\mathbb{E}_{A_n}(\mathbb{1}_2)$$

Since this is independent of our choice of $\psi^{(n)}_{\alpha\beta}$, the ground state of the infinite chain is unique.

Unique Ground State/Finite Correlation Length

Using the diagonalization of $\mathbb{E}_{\mathbb{1}_3}$, the respective linearities of the trace and $\mathbb{E}$, we can show, for two observables A, B located k sites away, that

$$\omega(AB) = \omega(A)\omega(B) + c\left(\frac{-1}{3}\right)^k$$

where the constant c only depends on A and B. This is enough to show that the correlation length for the AKLT model is $\frac{1}{\ln(3)}$.

Appendix

Here are some slides to help show
$\omega_n(A_1 \otimes A_2 \otimes \ldots A_n) = c_n \mathrm{Tr} P_\alpha \mathbb{E}_{A_1} \circ \mathbb{E}_{A_2} \circ \ldots \circ \mathbb{E}_{A_n}(P_\beta)$.

Let $|\beta\rangle \in \mathbb{C}^2$. Since $|\uparrow\rangle, |\downarrow\rangle \in \mathbb{C}^2$ are a basis for $\mathbb{C}^2$ there are elements $b_1, b_2 \in \mathbb{C}$ such that $|\beta\rangle = b_1|\uparrow\rangle + b_2|\downarrow\rangle$. Define the vector $|\widetilde{\beta}\rangle$ as:

$$|\widetilde{\beta}\rangle = b_1|\downarrow\rangle - b_2|\uparrow\rangle$$

It follows that $(\mathbb{1}_2 \otimes \langle\widetilde{\beta}|)\phi = |\beta\rangle$

Appendix

I will show the formula for the case $n = 1$. Consider any observable $A \in \mathcal{A}$. Then

$$\omega_1(A) = \frac{\langle W^*|\alpha\rangle \otimes |\beta\rangle,\ AW^*|\alpha\rangle \otimes |\beta\rangle\rangle}{\langle W^*|\alpha\rangle \otimes |\beta\rangle,\ W^*|\alpha\rangle \otimes |\beta\rangle\rangle}$$

Using the exercise $\langle\psi, \phi\rangle = \mathrm{Tr}(|\phi\rangle\langle\psi|)$ We rewrite $\omega_1(A)$ as

$$\omega_1(A) = \frac{\mathrm{Tr}(WAW^*|\alpha\rangle \otimes |\beta\rangle\langle\alpha| \otimes \langle\beta|)}{\mathrm{Tr}(|\alpha\rangle \otimes |\beta\rangle\langle\alpha| \otimes \langle\beta|)}$$

Let c be the constant $c = \mathrm{Tr}(|\alpha\rangle \otimes |\beta\rangle\langle\alpha| \otimes \langle\beta|)$ and from now on only consider the numerator.

Appendix

Let $|\phi\rangle$ be the singlet vector from $D^{(1/2)} \otimes D^{(1/2)} \cong D^{(1)} \otimes D^{(0)}$. Using the cyclicity of the trace we find:

$$
\begin{aligned}
&\mathrm{Tr}(WAW^*|\alpha\rangle \otimes |\beta\rangle\langle\alpha| \otimes \langle\beta|) \\
&= \mathrm{Tr}(WAW^*(|\alpha\rangle \otimes (\mathbb{1}_2 \otimes \langle\widetilde{\beta}|)|\phi\rangle)(\langle\alpha| \otimes (\langle\phi|\mathbb{1}_2 \otimes |\widetilde{\beta}\rangle))) \\
&= \frac{1}{2}\mathrm{Tr}[(WAW^* \otimes \mathbb{1}_2)(|\alpha\rangle \otimes \mathbb{1}_2 \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \langle\widetilde{\beta}|)(\mathbb{1}_2 \otimes |\phi\rangle) \\
&\qquad (\mathbb{1}_2 \otimes \langle\phi|)(\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes |\widetilde{\beta}\rangle)(\langle\alpha| \otimes \mathbb{1}_2 \otimes \mathbb{1}_2)] \\
&= \frac{1}{2}\mathrm{Tr}[(\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes |\widetilde{\beta}\rangle)(\langle\alpha| \otimes \langle\phi|)(W \otimes \mathbb{1}_2)(A \otimes \mathbb{1}_2) \\
&\qquad (W^* \otimes \mathbb{1}_2)(|\alpha\rangle \otimes |\phi\rangle)(\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \langle\widetilde{\beta}|)] \\
&= \frac{1}{2}\mathrm{Tr}[|\alpha\rangle\langle\alpha|V^*(A \otimes |\widetilde{\beta}\rangle\langle\widetilde{\beta}|)V)] = \frac{1}{2}\mathrm{Tr}[|\alpha\rangle\langle\alpha|\mathbb{E}_A(|\widetilde{\beta}\rangle\langle\widetilde{\beta}|)]
\end{aligned}
$$