Introduction to Quantum Spin Systems

Lecture 4: SU(2)

Bruno Nachtergaele

Mathematics, UC Davis

MAT290-25, CRN 30216, Winter 2011, 01/24/11

SU(2) and Spin

The physical observables of quantum spin with a 2-dimensional Hilbert space given by the three spin matrices

$$(S^1, S^2, S^3) = \frac{1}{2}(\sigma^1, \sigma^2, \sigma^3)$$
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

represent the three components of a vector quantity called angular momentum. This implies that the same physical system described in a rotated frame in 3-dimensional physical space should be related to the original one by a representation of the group of rotations in $\mathbb{R}^3$.

3

The superposition principle of quantum mechanics tells us to respect the linear structure of the Hilbert space of states, and since states are unit vectors, the transformation should also preserve the norm. Therefore we are looking for unitary transformations $U(R)$, for each 3-dimensional matrix $R \in SO(3)$ representing a rotation in $\mathbb{R}^3$ (we could also discuss general orthogonal transformations). For a group representation U we have

$$U(R_1 R_2) = U(R_1)U(R_2), R_1, R_2 \in SO(3)$$

Note however that expectation values would not see the phase-factor if U would be only a projective representation :

$$U(R_1 R_2) = e^{i\phi(R_1,R_2)}U(R_1)U(R_2), R_1, R_2 \in SO(3)$$

4

Equivalent to allowing for projective representations, is to consider representations of the universal covering group, in this case $SU(2)$. Since $SU(2)$ is compact, all its representations are equivalent to unitary representations. And these unitary representations are easily seen to be completely reducible, i.e., equivalent to a direct sum of irreducible representations. Therefore, what we need to study are the irreducible unitary representations of $SU(2)$, which turn out to be all finite-dimensional.

Ultimately, we are interested in systems of quantum spins, described by a Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$. Spin rotations are represented by tensor products of representations on the tensor factors. Hence, the need to study tensor products of representations.

SU(2) and its Lie algebra: SU(2) and $\mathfrak{su}_2$

The group $SU(2)$ is the group of unitary 2×2 complex matrices with determinant 1. Every such matrix can be uniquely written as

$$U(z, w) = \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$$

for $(z, w) \in \mathbb{C}^2$, with the condition that $|z|^2 + |w|^2 = 1$. In other words, $SU(2)$ is topologically equivalent to the unit sphere in $\mathbb{C}^2$, which is the same as the real 3-sphere. $SU(2)$ is a real Lie group, meaning it is a group with a compatible structure of a real manifold. This can be made explicit by writing, e.g., $z = e^{i\alpha} \cos\gamma, w = e^{i\beta} \sin\gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}$. The group structure, together with the fact that the dependence of the group elements on the coordinates of the manifold can be taken to be analytic, has some very interesting implications.

Define

$$U(\theta_1, \theta_2, \theta_3) = e^{i(\theta_1\sigma^1 + \theta_2\sigma^2 + \theta_3\sigma^3)}.$$

Then $U(\theta_1, \theta_2, \theta_3) \in SU(2)$ for $\theta \equiv (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. and

$$\frac{\partial}{\partial\theta_j} U(\theta)|_{\theta=0} = i\sigma^j, \quad j = 1, 2, 3\,.$$

A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear form $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a Lie bracket, satisfying

1. $[Y, X] = -[X, Y]$,
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The Lie algebra $\mathfrak{su}_2$ is defined as the tangent space to $SU(2)$ at the identity.

The previous calculation shows that the spin-$1/2$ matrices are a basis of $\mathfrak{su}_2$:

$$S^1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

The spin matrices satisfy the commutation relations $[S^1, S^2] = iS^3$, $[S^2, S^3] = iS^1$, $[S^3, S^1] = iS^2$.
The (irreducible) hermitian representations of $\mathfrak{su}_2$ are in one-to-one correspondence with the (irreducible) unitary representations of $SU(2)$. This correspondence respects tensor products if we define the tensor product X of two Lie algebra representations X_1 and X_2 by

$$X = X_1 \otimes \mathbb{1} + \mathbb{1} \otimes X_2$$

The Power of Representation Theory

Let U be a unitary representation of a group G on a Hilbert space $\mathcal{H}$. And suppose $H = H^* \in \mathcal{B}(\mathcal{H})$ (a Hamiltonian) commutes with this representation:

$$U(g)H = HU(g), \quad \text{for all } g \in G.$$

Consider any irreducible subrepresentation $V(g) = PU(g)P$, where P is an orthogonal projection onto a minimal invariant subspace $\mathcal{H}_0 = \mathrm{ran}P$ for $\{U(g) \mid g \in G\}$. Then, if $\mathcal{H}_0$ contains an eigenvector of H, we must have $PHP = \lambda P$, for $\lambda \in \mathbb{R}$ the corresponding eigenvalue.

The irreps of $SU(2)$

In standard Physics notation, the irreps of $\mathfrak{su}_2$ (and hence $SU(2)$) are specified by the representation of the three spin matrices. It turns out that, up to unitary equivalence, there is exactly one unitary irreducible representation of dimension d, for $d \geq 1$. The d-dimensional irrep is called the spin-S representation with $d = 2S + 1$ ($S = 0, 1/2, 1, 3/2, 2, \ldots$). The matrices are conventionally given with respect to an orthonormal basis of eigenvectors of S^3, which is denoted by $|m\rangle\,, m = S, S-1, \cdots, -S$, where the index m is also the corresponding eigenvalue.

$$S^1 = \frac{S^+ + S^-}{2}, \quad S^2 = \frac{S^+ - S^-}{2i},$$

with

$$S^+ |m\rangle = c_{m+1} |m+1\rangle \ , \quad S^- |m\rangle = c_m |m-1\rangle \ ,$$

where

$$c_m = \sqrt{S(S+1) - m(m-1)} = \sqrt{(S+m)(S+1-m)}\,.$$

Note that $c_{S+1} = c_{-S} = 0$.

$$\begin{aligned} S^+ &= \begin{pmatrix} 0 & c_S & & & \\ & 0 & c_{S-1} & & \\ & & \ddots & \ddots & \\ & & & 0 & c_{-S+1} \\ & & & & 0 \end{pmatrix} \\ S^- &= (S^+)^* \end{aligned}$$

Or equivalently: $(S^{\pm})_{m,n} = c_m \delta_{m,n\pm 1}$. The $SU(2)$ commutation realtions are $[S^+, S^-] = 2S^3$ and one can quickly derive

$$\begin{aligned} [S^3, S^+] \, |m\rangle &= ((m+1)c_{m+1} - mc_{m+1}) \, |m+1\rangle \\ &= S^+ \, |m\rangle \\ \Rightarrow [S^3, S^+] &= S^+ \end{aligned}$$

and similarly, $[S^3, S^-] = -S^-$. These relations are easily seen to be equivalent to

$$[S^\alpha, S^\beta] = i \sum_\gamma \varepsilon_{\alpha\beta\gamma} S^\gamma \,, \qquad \begin{array}{l} \alpha, \beta, \gamma = 1, 2, 3 \\ \varepsilon_{123} = 1 \text{ and totally antisymmetric} \end{array}$$

The Clebsch-Gordan series

For $j = 0, 1/2, 1, \ldots$, the $(2j+1)$-dimensional irrep of $SU(2)$ obtained by exponentiating the representation of $\mathfrak{su}_2$ given by the spin matrices we defined above, is often denoted by $D^{(j)}$. The Clebsch-Gordan series gives the decomposition of the tensor product of two irreps as a direct sum of irreps:

$$D^{(j_1)} \otimes D^{(j_2)} \cong D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \cdots \oplus D^{(|j_1-j_2|)} \,.$$

The Clebsch-Gordan coefficients give the expansion of bases of each of the irreps on the RHS in terms of the standard tensor product bases of the LHS:

$$|j, m\rangle = \sum_{m_1, m_2} C(j_1, j_2; m_1, m_2 \mid j, m) \, |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

for $j = j_1 + j_2, j_1 + j_2 - 1, \ldots |j_1 - j_2|$.

The Heisenberg Model, Cont'd

For $x, y \in \mathbb{Z}^d$, let $|x-y|$ denote the lattice distance between x and y. x and y are called nearest neighbors if $|x-y|=1$. Let $\mathcal{H}_x = \mathbb{C}^{2S+1}$, for all x.

The Hamiltonians of the spin-S (homogeneous and isotropic) Heisenberg ferromagnet on Λ is given by

$$H_\Lambda = -\sum_{x,y\in\Lambda,|x-y|=1} \mathbf{S}_x \cdot \mathbf{S}_y \qquad (1)$$

where $\mathbf{S} = (S^1, S^2, S^3)$ are the spin-S matrices.

$$[\mathbf{S}_x \cdot \mathbf{S}_y, D_x^{(S)} \otimes D_y^{(S)}] = 0$$

It is straightforward to verify that the basis in which the representation are block-diagonal (defined by the Clebsch-Gordan series) diagonalizes $\mathbf{S}_x \cdot \mathbf{S}_y$ with eigenvalues

$$\lambda_j = \frac{1}{2}\left(j(j+1) - 2S(S+1)\right), \quad j = 0, 1, \ldots, 2S.$$

Back to the spin $1/2-$ Heisenberg Model

The Hamiltonian is

$$H_\Lambda = -J \sum_{|x-y|=1,x,y\in\Lambda} \mathbf{S}_x \cdot \mathbf{S}_y$$

with $J > 0$ ($J < 0$) corresponding to the (anti-)ferromagnet. It is useful the introduce $\sigma^\pm$ such that

$$\sigma^1 = \sigma^+ + \sigma^-, \quad \sigma^2 = i(\sigma^- - \sigma^+).$$

Then

$$\mathbf{S}_x \cdot \mathbf{S}_y = \frac{1}{2}(\sigma_x^+\sigma_y^- + \sigma_x^-\sigma_y^+) + \frac{1}{4}\sigma_x^3\sigma_y^3 = \frac{1}{2}t_{xy} - \frac{1}{4}\mathbb{1}.$$

where $t_{xy} \in \mathcal{A}_{\{x,y\}}$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ as the swap of the two tensor factors: $t(u \otimes v) = v \otimes u$.

Clearly the ground states of the ferromagnetic model are the eigenvectors with smallest eigenvalue of the operator

$$\tilde{H}_\Lambda = -\sum_{|x-y|=1, x,y\in\Lambda} t_{xy}$$

Consider Λ as a graph with N vertices $x \in \Lambda$ and E edges (xy) given by the nearest neighbor pairs.

Theorem

Suppose Λ considered as a graph is connected. Then, the smallest eigenvalue of $\tilde{H}_\Lambda$ is $-E$ and its eigenspace has dimension $N+1$ and consist of all vectors symmetric under permutations of the vertices.

Proof

Let $\{e_-, e_+\}$ be an o.n. basis of $\mathbb{C}^2$, and $\lambda_0(A)$ the smallest eigenvalue of a matrix A.

Observation 1)

For any Hermitian operators, $\lambda_0(A+B) \geq \lambda_0(A) + \lambda_0(B)$. (For any hermitian matrix the lowest eigenvalue satisfies the variational principle $\lambda_0(A) = \inf_{\psi\neq 0}\langle A\rangle_\phi$. Thus $\lambda_0(A+B) = \inf_{\psi\neq 0}\langle A+B\rangle_\psi$, while for any ψ, $\langle A+B\rangle_\psi = \langle A\rangle_\psi + \langle B\rangle_\psi \geq \lambda_0(A) + \lambda_0(B)$, which proves the observation.) Apply this to $\tilde{H}$ to obtain $\lambda_0(\tilde{H}) \geq -E$.

Observation 2)

$$\tilde{H}(\bigotimes_{x\in\Lambda} e_+) = -E \bigotimes_{x\in\Lambda} e_+$$

which implies that $\lambda_0(H)$ does equal $-E$.

17

Observation 3) If $\lambda_0(A+B) = \lambda_0(A) + \lambda_0(B)$ then $(A+B)\phi = \lambda_0(A+B)$ implies $A\phi = \lambda_0(A)\phi$ and $B\phi = \lambda_0(B)\phi$. (For a hermitian matrix A, ψ is an eigenvector for $\lambda_0(A)$ iff $\langle A\rangle_\psi = \lambda_0(A)$. Now suppose $\lambda_0(A+B) = \lambda_0(A) + \lambda_0(B)$ and ϕ is a ground state for $A+B$. Then

$$\langle A\rangle_\phi + \langle B\rangle_\phi = \langle A+B\rangle_\phi = \lambda_0(A+B) = \lambda_0(A) + \lambda_0(B)$$

Also, $\langle A\rangle_\phi \geq \lambda_0(A)$ and $\langle B\rangle_\phi \geq \lambda_0(B)$. So we have $\langle A\rangle_\phi = \lambda_0(A)$ and $\langle B\rangle_\phi = \lambda_0(B)$. And this implies $A\phi = \lambda_0(A)\phi$ and $B\phi = \lambda_0(B)\phi$.) Apply this to determine $H\phi = -E\phi$ only if $t_{xy}\phi = \phi$ for all $(xy) \in \Lambda$.

18

Observation 4) If Λ is a connected graph then any permutation of its vertices, π, can be written as a product of transpositions τ_{xy} where (xy) is an edge in the graph. Apply this to deduce that for any ground state ϕ, $U_\pi\phi = \phi$ for all permutations π, where U_π is the unitary determined by its action on simple tensors:

$$U_\pi(v_1 \otimes \cdots \otimes v_N) = v_{\pi_1} \otimes \cdots \otimes v_{\pi_N}$$

So, all ground states have to be permutation-invariant vectors, and vice-versa if $U_\pi \phi = \phi$ for all π then $t_{xy}\phi = \phi$ for all x, y (because $t_{xy} = U_{\tau_{xy}}$). An o.n. basis of permutation invariant vectors is given by

$$\psi_k = \frac{1}{N!} \sum_{\pi \in \mathrm{Perm}(\Lambda)} U_\pi ((\underbrace{e_- \otimes \cdots \otimes e_-}_{k}) \otimes (\underbrace{e_+ \otimes \cdots \otimes e_+}_{N-k}))$$

for $k = 0, 1, \cdots, N$.
This is also the maximum spin irreducible representation contained in $\otimes_{x\in\Lambda}\mathcal{D}^{(1/2)}$. More about that later.

(Let $\mathcal{G}$ be the group generated by $\{\tau_{xy} : (xy) \in \Lambda\}$. Prove by induction that $\tau_{xy} \in \mathcal{G}$ for all i and j, vertices in Λ. Induction is on

$$d(x, y) = \min\{n : (xx_2), (x_2x_3), \ldots, (x_{n-1}y) \in \Lambda\}$$

For $d(x, y) = 0$ it is trivial that $\tau_{xx} = id \in \mathcal{G}$. For the induction step assume that $\tau_{xy} \in \mathcal{G}$ whenever $d(x, y) \leq n$. If $d(x, y) = n + 1$ let $x_2, \cdots, x_n$ be a sequence s.t. $(xx_2), \ldots, (x_ny) \in \Lambda$. Then $\tau_{xx_n}, \tau_{x_ny} \in \mathcal{G}$ by the induction hypothesis. So $\tau_{xy} = \tau_{x_ny}\tau_{xx_n}\tau_{x_ny} \in \mathcal{G}$. Since the permutation group is generated by all transpositions,
$\mathcal{G} = \mathrm{Perm}(\{x : x \in \Lambda\})$.)