Do any 5 from the 6 problems. Please, cross out the problem you do not want to have graded.

This test is closed book. Whenever you use a known theorem, either use a common name to identify it uniquely, or state it.

The write-up of your proofs should be explicit, precise, and reasonably complete.

Good luck!

1. For $a \geq 0$, and $b \in \mathbb{R}$, define $f_{a,b} : [-1,1] \to \mathbb{R}$, by

$$f_{a,b}(x) = e^{-a(x-b)^2}, \quad \text{for all } x \in [-1,1].$$

Clearly, $f_{a,b} \in C([-1,1])$, for all $a \geq 0$, and $b \in \mathbb{R}$.

a) Find the functions $g_b$, $b \in \mathbb{R}$, and $h_{a}^\pm$, $a \geq 0$, defined by the following pointwise limits:

$$g_b(x) = \lim_{a \to \infty} f_{a,b}(x)$$

$$h_{a}^\pm(x) = \lim_{b \to \pm \infty} f_{a,b}(x)$$
b) Consider the set

\[ A_0 = \left\{ f_{a,b} \mid a > 0, b \in \mathbb{R} \right\} \]

Is \( A_0 \) a compact subset of \( (C([-1,1]), \| \cdot \|_{sup}) \)? Prove or disprove.

c) Define

\[ A_1 = \left\{ f_{a,b} \mid 0 \leq a \leq 1, b \in \mathbb{R} \right\} \]

Is \( A_1 \) a compact subset of \( (C([-1,1]), \| \cdot \|_{sup}) \)? Prove or disprove.
d) Is $A_0$ a pre-compact subset of $(C([-1,1]), \| \cdot \|_{sup})$?

f) Is $A_1$ a pre-compact subset of $(C([-1,1]), \| \cdot \|_{sup})$?

2. Let $X \neq \{0\}$ be a normed linear space, and let $\mathcal{B}(X)$ be the space of all bounded linear maps $X \to X$. Recall that the uniform norm on $\mathcal{B}(X)$ is defined as follows:

$$
\|T\| = \inf \{ M \geq 0 \mid \|Tx\| \leq M\|x\|, \text{ for all } x \in X \}, \text{ for all } T \in \mathcal{B}(X) \quad (i)
$$

Prove that the following are three equivalent expressions for $\|T\|$:

$$
\|T\| = \sup \{ \|Tx\| \mid x \in X, \|x\| \leq 1 \} \quad (ii)
$$

$$
\|T\| = \sup \{ \|Tx\| \mid x \in X, \|x\| = 1 \} \quad (iii)
$$

$$
\|T\| = \sup \left\{ \left\| \frac{Tx}{\|x\|} \right\| \mid x \in X, x \neq 0 \right\} \quad (iv)
$$
3. In the study of a well-known model for a gas of particles confined to move on a straight line, one encounters the following equation for a function $f : \mathbb{R} \to \mathbb{R}$, depending on a parameter $a, 0 < a < +\infty$:

$$f(t) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (s - t)^2} f(s) \, ds$$

a) Prove that, for all $a \in (0, +\infty)$, this equation has exactly one continuous solution $f_a : \mathbb{R} \to \mathbb{R}.$
b) Prove that \( f_\alpha(x) \geq 0 \), for all \( x \in \mathbb{R} \), where \( f_\alpha \) is the unique continuous solution found in a).

4. For any \( f \in C([0,1]) \), and \( \alpha \in (0,1) \), define \( H_\alpha(f) \) by

\[
H_\alpha(f) = \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \mid 0 \leq x < y \leq 1 \right\}
\]

Let \( (f_n) \) be a sequence in \( C([0,1]) \), converging to \( f \) in the uniform norm. Show that if for some \( \alpha \in (0,1) \), and some constant \( C \geq 0 \), \( H_\alpha(f) \leq C \), for all \( n = 1,2,\ldots \), then \( H_\alpha(f) \leq C \).
5. Let $X$ be a Banach space.

a) Prove that, for all $A \in \mathcal{B}(X)$, the following series converges:

$$\exp A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$
b) Prove that $\exp : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$, as defined in a), is continuous.
6. Let \( X \) be a Banach space. For \( r \geq 0 \) and \( x_0 \in X \), the sphere with radius \( r \) and center \( x_0 \), denoted by \( S_r(x_0) \), is defined by

\[
S_r(x_0) = \{ x \in X \mid \|x - x_0\| = r \}
\]

a) Prove that if \( X \) is a Hilbert space, \( S_r(x_0) \) does not contain any linear segment. More precisely, show that, for any linear segment \( L \) of the form

\[
L = \{ x + ty \mid t \in (-\delta, \delta) \}
\]

with \( x, y \in X \), \( y \neq 0 \), and \( \delta > 0 \), we have that

\[
L \cap S_r(x_0) \text{ contains at most one element.}
\]

(Hint: show that, for any \( z, y \in X \), if \( \|z + ty\| = r \), for all \( t \in (-\delta, \delta) \) for some \( \delta > 0 \), then \( y = 0 \).)

b) Show with an example that in some Banach spaces (necessarily not Hilbert spaces), every sphere of non-zero radius contains flat parts, e.g., linear segments.