Problem 1. Let \((x_n)_{n=1}^\infty\) be a sequence in a Banach space. Suppose that \(\|x_n\| \leq a_n\), where \((a_n)_{n=1}^\infty\) is a sequence of nonnegative real numbers such that \(\sum_{n=1}^\infty a_n\) converges. Prove that \(\sum_{n=1}^\infty x_n\) converges.

Problem 2. Let \(X\) be the linear space of continuous functions \(f : [0, 1] \rightarrow \mathbb{R}\) with the norm
\[
\|f\| = \left( \int_0^1 |f(x)|^2 \, dx \right)^{1/2}.
\]
(a) Prove that \((X, \| \cdot \|)\) is not complete.
(b) Describe the completion (no proof required).

Problem 3. Let \(k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}\) be a continuous function, and define the integral operator \(K : C([0, 1]) \rightarrow C([0, 1])\) by
\[
(Kf)(x) = \int_0^1 k(x, y)f(y) \, dy.
\]
(a) Prove that
\[
E = \{Kf \mid f \in C([0, 1]), \|f\| \leq 1\}
\]
is an equicontinuous set in \(C([0, 1])\), where \(\| \cdot \|\) denotes the sup norm.
(b) If \((f_n)\) is a sequence in \(C([0, 1])\) with \(\|f_n\| \leq 1\), prove that there is a subsequence \((f_{n_j})\) such that \((Kf_{n_j})\) converges uniformly.

Problem 4. Define a function \(L : C([0, 1]) \rightarrow \mathbb{R}\) by
\[
L(f) = \int_0^{1/2} f(x) \, dx - \int_{1/2}^1 f(x) \, dx.
\]
(a) Prove that \(L\) is continuous. (Here, \(C([0, 1])\) is equipped with the sup norm.)
(b) Let \(S_n\) denote the set of polynomials \(p\) of degree less than or equal to \(n\) with \(\|p\| = 1\). Prove that \(L\) attains its maximum on \(S_n\).
(c) Let \(S\) denote the set of continuous functions \(f\) on \([0, 1]\) with \(\|f\| = 1\). Prove that
\[
\sup_{f \in S} L(f) = 1,
\]
but that \(L\) does not attain its supremum on \(S\).