Problem 1. Let \( n \geq 1 \) and consider \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \). Let \( C_0 \subset \mathbb{R}^n \) denote the essential domain of \( f \), i.e., the set of points where \( f \) is finite. Recall the definition of \( f^* \), the Legendre transform of \( f \):

\[
f^* = \sup_{x \in \mathbb{R}^n} (\langle y, x \rangle - f(x)) .
\]

Define the epigraph of \( f \), \( \text{epi}(f) \), to be the set

\[
\text{epi}(f) = \{(x, r) \in C_0 \times \mathbb{R} \mid r \geq f(x)\}.
\]

a) Suppose \( C_0 \) is convex. Then, show that \( f \) is convex iff \( \text{epi}(f) \) is a convex subset of \( \mathbb{R}^{n+1} \).

b) Prove that for any \( f \), we have \( \text{epi}(f^{**}) = \overline{\text{co epi}(f)} \), where \( \overline{\text{co}} \) denotes the closed convex hull.

c) Show that if \( f \) is convex, then

\[
(f^*)^*(x) = f(x),
\]

for all \( x \) in the interior of \( C_0 \). (Hint: note that \( \overline{\text{co epi}(f)} \) is the intersection of a family of closed halfspaces.)

d) Let \( n = 1 \) and \( f(x) = |x| \). Find \( f^* \).

Problem 2. Consider the one-dimensional Ising model defined on intervals of the form \([-L, L] \subset \mathbb{Z}\) with Hamiltonians \( H^b_L \) defined by

\[
H^b_L = -J \sum_{x=-L}^{L-1} \sigma_x \sigma_{x+1} + b_L
\]

where, for each \( L \geq 1 \), \( b_L \) is a function of the boundary spin variables \( \sigma_{-L} \) and \( \sigma_L \). Fix any inverse temperature \( \beta \in \mathbb{R} \). Prove that the sequence Gibbs states at inverse temperature \( \beta \), \( \omega_L \) on \( C(\Omega_{[-L,L]}) \), determined by \( H^b_L \), converges (in the weak-* sense) to a translation invariant state on \( C(\Omega_{\mathbb{Z}}) \), which depends on \( J \) and \( \beta \), but not on \( b_L \).

Problem 3. Let \( \gamma \) be a non-selfintersecting closed path consisting of nearest neighbor bonds in \( \mathbb{Z}^2 \) (i.e., a simple contour), and denote by \( V(\gamma) \) the lattice sites enclosed by \( \gamma \), and by \( l(\gamma) \) the length of \( \gamma \) (i.e., the number of bonds).

a) Prove the following bound:

\[
|V(\gamma)| \leq \frac{1}{16} l(\gamma)^2.
\]
b) Consider a finite subset $\Lambda \subset \mathbb{Z}^2$, and let $l = 4, 6, 8, \ldots$. Let $M_\Lambda(l)$ denote the number of simple contours $\gamma$ of length $l$ contained in $\Lambda$. Prove the bound

$$M_\Lambda(l) \leq 3^{l-1} |\Lambda|,$$

for all $l = 4, 6, 8, \ldots$.

c) (Optional) Show that there exists $c > 1$ such that for all $l = 4, 6, 8, \ldots$, there exists $L_l$ such that for all $\Lambda = [1, L]^2$, with $L \geq L_l$ one has the bound

$$M_\Lambda(l) \geq L^2 c^l.$$

**Problem 4.** Fix $J > 0$ and $\beta \geq 0$. Prove that the Gibbs states at inverse temperature $\beta$ of the $d$-dimensional translation invariant Ising model defined on $[-L, L]^d \subset \mathbb{Z}^d$ with periodic boundary conditions (i.e., defined on tori), converge in the thermodynamic limit $L \to \infty$.

**Problem 5.** Let $\omega_{d,\beta}^+$ be the limiting Gibbs state of the translation invariant ferromagnetic Ising model on $\mathbb{Z}^d$ with coupling constant $J = 1$, at inverse temperature $\beta \geq 0$. Define the critical inverse temperature, $\beta_c$, by

$$\beta_c = \sup \{\beta \geq 0 \mid \omega_{d,\beta}^+(\sigma_0) = 0\}.$$

Prove that $\beta_c(d)$ is a monotone decreasing function of the dimension $d$. 