Statistical Mechanics, Math 266: Week 3 Notes

January 19 and 21, 2010

1 Phase Transitions and Spontaneous Symmetry Breaking

Consider the d-dimensional Ising model:

$$\Lambda \subseteq \mathbb{Z}^d \text{ e.g., } \Lambda = [1, L]^d \tag{1}$$

$$H_\Lambda = -J \sum_{\substack{|x-y|=1 \\ x,y\in\Lambda}} \sigma_x \sigma_y \tag{2}$$

As before, we will assume that the model is ferromagnetic, so $J > 0$. The Hamiltonian H_Λ exhibits a spin flip symmetry which takes $\sigma_x \to -\sigma_x$ Many fundamental models have symmetries and many interesting phase transitions are accompanied by symmetry breaking. More precisely, let $F : \Omega \to \Omega$ be defined by

$$F(\eta) = -\eta \tag{3}$$

Clearly $H_\Lambda \circ F = H_\Lambda$. It follows that the equilibrium state

$$\omega_\Lambda(f) = \frac{\sum_\eta f(\eta) e^{-\beta H_\Lambda(\eta)}}{\sum_\eta e^{-\beta H_\Lambda(\eta)}} \tag{4}$$

is also F-symmetric, meaning

$$\omega_\Lambda(f \circ F) = \omega_\Lambda(f) \text{ for all } f \in C(\Omega_\Lambda) \tag{5}$$

In particular, we have for all $x \in \Lambda$,

$$\omega_\Lambda(\sigma_x) = \omega_\Lambda(\sigma_x \circ F) = -\omega_\Lambda(\sigma_x) \tag{6}$$

$$\text{and hence } \omega_\Lambda(\sigma_x) = 0 \text{ for all } x \in \Lambda \tag{7}$$

Taking the thermodynamic limit does not change this,

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \omega_\Lambda(\sigma_x) = 0 \tag{8}$$

All of the thermodynamics is contained in the function $f(\beta)$, the free energy density

$$-\beta f(\beta) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda(\beta) \tag{9}$$

$$Z_\Lambda(\beta) = \sum_{\eta \in \Omega_\Lambda} e^{-\beta H_\Lambda(\eta)} \tag{10}$$

It is easy to see that boundary conditions do not affect f, so in the thermodynamic limit, we obtain the same thermodynamics.

Now define the boundary of Λ,

$$\partial\Lambda = \{x \in \Lambda | \text{ there exists } y \in \mathbb{Z}^d \cup \Lambda^c \text{ and } |x - y| = 1\} \tag{11}$$

Then consider $b_\Lambda \in C(\Omega_{\partial\Lambda})$ and suppose that $\|b_\Lambda\|_{\sup} \leq B|\partial\Lambda|$, so that b_Λ is uniformly bounded with B some fixed constant. For some sequence of boundary terms, we may find that

$$\omega^b = \lim_\Lambda \omega_\Lambda^b \text{ exists} \tag{12}$$

$$\omega_\Lambda^{b_\Lambda}(f) = \frac{1}{Z_\Lambda^{b_\Lambda}(\beta)} \sum_{\eta \in \Omega_\Lambda} f(\eta) e^{-\beta H_\Lambda^{b_\Lambda}(\eta)} \tag{13}$$

$$\text{where } H_\Lambda^{b_\Lambda} = H_\Lambda + b_\Lambda \tag{14}$$

Then,

$$e^{-B\beta|\partial\Lambda|} Z_\Lambda \leq Z_\Lambda^{b_\Lambda} \leq Z_\Lambda e^{B\beta|\partial\Lambda|} \tag{15}$$

So that as long as we have $\frac{|\partial\Lambda|}{|\Lambda|} \to 0$, we will have that $f^b(\beta) = f(\beta)$. It is therefore reasonable to assume that $\omega^b = \lim_\Lambda \omega_\Lambda^{b_\Lambda}$. If this limit exists, it will also describe the equilibrium of the thermodynamic system. (We will make this precise and rigorous when we study characteristics of equilibrium later.) Under quite general conditions, one can show that for some $\beta_c > 0$, ω^b is independent of b for all $0 \leq \beta \leq \beta_c$. But it often happens that there is some dependence on the boundary condition, b, if β is large enough. Before doing anything more general, we will show that this happens for the d-dimensional Ising model.

2 The Peierls Argument

We will consider the particular boundary term leading to what is called + boundary conditions,

$$b_\Lambda = -J \sum_{\substack{x \in \partial\Lambda \\ y \in \Lambda^c, |x-y|=1}} \sigma_x \cdot 1 \tag{16}$$

as if the spin at $y \in \Lambda^c$ are all fixed to be $+1$. $-$ boundary conditions are completely analogous. Let's assume that the following limit exists:

$$\omega^+ = \lim_\Lambda \omega_\Lambda^+ \tag{17}$$

$$= \lim_\Lambda \omega_\Lambda^{b_\Lambda} \tag{18}$$

Note that if we were free to interchange limits, it would be rather trivial to show that $\lim_{\beta\to\infty} m(\beta) = 1$ since $\lim_{\beta\to\infty} \omega_\Lambda^+(\sigma_x) = 1$ for all finite $\Lambda \subseteq \mathbb{Z}^d$, for all $d \geq 1$.

Theorem 2.1. *Let $J > 0$ and $d = 2$. Then,*

1. *There exists $\beta_1 > 0$ such that for all $\beta > \beta_1$,*

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} \omega_\Lambda^+(\sigma_x) = m_\Lambda(\beta) > 0 \tag{19}$$

   and $\lim_{\beta\to\infty} m_\Lambda(\beta) = 1$ uniformly in Λ.

2. *There exists $\beta_2 > 0$ such that for all $\beta > \beta_2$ and for all $x \in \Lambda$,*

$$\omega_\Lambda^+(\sigma_x) = m_x(\beta) > 0 \tag{20}$$

   and $\lim_{\beta\to\infty} m_x(\beta) = 1$ uniformly in x.

In fact, we obtain bounds of the form

$$0 \leq 1 - m_x(\beta) \leq 216e^{-8J\beta} \tag{21}$$

$$0 \leq 1 - m_\Lambda(\beta) \leq 216e^{-8J\beta} \tag{22}$$

For sufficiently large β.

Remark 2.1. *Onsager obtained an exact solution of the free energy density if the 2-dimensional Ising model frem which it follows that $\beta_c = \frac{\log(1+\sqrt{2})}{2J}$.*

To prove the theorem we will use the contour description of

$$\Omega_\Lambda^+ = \{\eta \in \Omega_{\Lambda\cup\partial(\Lambda^c)} | \eta \restriction_{\Lambda^c} = +1\} \tag{23}$$

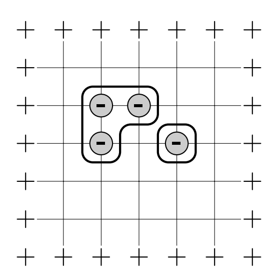

Figure 1: Configuration space as described by contours.

There is a one-to-one correspondence between configurations and configurations of contours.

$$\sigma \in \Omega_\Lambda^+ \longleftrightarrow \{\gamma_1, \ldots, \gamma_n\} = \Gamma(\sigma) \tag{24}$$

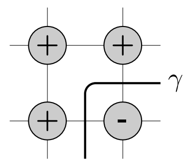

Figure 2: A dual edge is in γ if the corresponding edge has a $+-$ pair.

Here, $\Gamma(\sigma)$ is a configuration of closed, non-intersecting paths (where we ignore corner intersection) in the dual lattice. We define the support of a configuration,

$$\operatorname{supp}\Gamma = \bigcup_{i=1}^{n} \operatorname{supp}\gamma_i = \{(x,y) \in \mathbb{Z}^2 | \sigma_x \sigma_y = -1\} \tag{25}$$

and this allows us to define the length of a configuration of contours as

$$\ell(\Gamma) = |\operatorname{supp}\Gamma| = \sum_{i=1}^{n} \ell(\gamma_i) \tag{26}$$

Now, we wish to rewrite the Hamiltonian of the Ising model in terms of contours.

$$H_\Lambda^+(\sigma) = -J\{\#\{(x,y) \in \mathbb{Z}^2 | \sigma_x\sigma_y = +1\}\} + J\{\#\{(x,y) \in \mathbb{Z}^2 | \sigma_x\sigma_y = -1\}\} \tag{27}$$

$$= -J|B(\Lambda)| + 2J\ell(\Gamma) \tag{28}$$

where $|B(\Lambda)|$ is equal to the number of edges in Λ. Here, we can think of energy as being proportional to the length of the contours. Since the configurations $\Gamma(\sigma)$ that we consider consist of compatible, closed, non-intersecting paths, we define $V(\gamma)$ to be the vertices enclosed by a given contour γ.

Lemma 2.1. *For all γ such that $\ell(\gamma) < \infty$,*

$$|V(\gamma)| \leq \frac{1}{16}\ell(\gamma)^2 \tag{29}$$

Proof. Left to reader. See homework. □

Lemma 2.2. *Let $\Lambda \subseteq \mathbb{Z}^2$ and $\ell = 4, 6, 8, \ldots$ Define $M_\Lambda(\ell) = \#$ of distinct simple contours of length ℓ within Λ (with '+' boundary conditions). Then*

$$M_\Lambda(\ell) \leq 3^{\ell-1}|\Lambda| \tag{30}$$

Proof. Left to reader. See homework. □

We know what $\mathbb{P}^+_\Lambda(\sigma)$ is, and this allows us to make sense of $\mathbb{P}(\Gamma)$. Now, for a given contour γ, we define $\mathbb{P}^+_\Lambda(\gamma)$ to be the probability that γ occurs. Specifically, it is the event that contains all configurations Γ that have γ in it. Explicitly,

$$\mathbb{P}^+_\Lambda(\gamma) = \frac{\sum_{\substack{\sigma\in\Omega^+_\Lambda \\ \gamma\in\Gamma(\sigma)}} e^{-\beta H^+_\Lambda(\sigma)}}{\sum_{\sigma\in\Omega^+_\Lambda} e^{-\beta H^+_\Lambda(\sigma)}} \tag{31}$$

Lemma 2.3 (Peierls Estimate)**.**

$$\mathbb{P}^+_\Lambda(\gamma) \leq e^{-2J\beta\ell(\gamma)} \tag{32}$$

Proof. For all σ such that $\gamma \in \Gamma(\sigma)$, for some fixed γ define σ^* as the unique configuration such that $\Gamma(\sigma^*) = \Gamma(\sigma)\backslash\{\gamma\}$. Explicitly, σ^* is obtained by flipping all spins located at $x \in V(\gamma)$. Recall that $H^+_\Lambda(\sigma) = -JB(\Lambda) + 2J\ell(\gamma)$, and therefore

$$H^+_\Lambda(\sigma) - H^+_\Lambda(\sigma^*) = 2J\ell(\Gamma) \tag{33}$$

and

$$\mathbb{P}^+_\Lambda(\gamma) = \frac{\sum_{\substack{\sigma\in\Omega^+_\Lambda \\ \gamma\in\Gamma(\sigma)}} e^{-\beta H^+_\Lambda(\sigma)}}{\sum_{\sigma\in\Omega^+_\Lambda} e^{-\beta H^+_\Lambda(\sigma)}} \tag{34}$$

$$\leq \frac{\sum_{\substack{\sigma\in\Omega^+_\Lambda \\ \gamma\in\Gamma(\sigma)}} e^{-\beta H^+_\Lambda(\sigma)}}{\sum_{\substack{\sigma\in\Omega^+_\Lambda \\ \gamma\in\Gamma(\sigma)}} e^{-\beta H^+_\Lambda(\sigma^*)}} \tag{35}$$

$$\leq e^{-2\beta J\ell(\gamma)} \tag{36}$$

□

Proof of Theorem 2.1. We will estimate

$$0 \leq 1 - \omega^+_\Lambda(\sigma_x) = \omega^+_\Lambda(1 - \sigma_x) \tag{37}$$

Observe that $1 - \sigma_x$ takes the values 0 and 2. If $1 - \sigma_x = 2$, then there exists a $\gamma \in \Gamma(\eta)$ such that $x \in V(\gamma)$. Denote by $\gamma^*(\sigma)$ the *first* contour you meet starting at x. Then

$$1-\omega_\Lambda^+(\sigma_x) \le \frac{2\sum_{\gamma,x\in V(\Gamma)}\sum_{\sigma,\gamma^*(\sigma)=\gamma} e^{-\beta H_\Lambda^+(\sigma)}}{\sum_\sigma e^{-\beta H_\Lambda^+(\sigma)}} \tag{38}$$

$$\le 2\sum_{\gamma,x\in V(\gamma)} \frac{\sum_{\sigma,\gamma\in\Gamma(\sigma)} e^{-\beta H_\Lambda^+(\sigma)}}{\sum_\sigma e^{-\beta H_\Lambda^+(\sigma)}} \tag{39}$$

$$= 2\sum_{\gamma,x\in V(\gamma)} \mathbb{P}_\Lambda^+(\gamma) \tag{40}$$

$$\le 2\sum_{\gamma,x\in V(\gamma)} e^{-2\beta J\ell(\gamma)} \tag{41}$$

where (39) comes from the fact that $1-\omega_\Lambda^+$ vanishes if it is enclosed by an even number of contours, and (41) follows from Lemma 2.3. From here, Parts 1 and 2 of Theorem 2.1 proceed only slightly differently, and so we only present the proof of Part 2. We rewrite the inequality (41) in a more suggestive way.

$$1-\omega_\Lambda^+(\sigma_x) \le 2\sum_{\ell=4,6,8,\ldots}\sum_{\substack{\gamma,x\in V(\gamma)\\ \ell(\gamma)=\ell}} e^{-2\beta J\ell} \tag{42}$$

$$\le 2\sum_{\ell=4,6,8,\ldots} \ell^2 3^\ell e^{-2\beta J\ell} \tag{43}$$

$$\le 16(3e^{-2\beta J})^2 \text{ where } 3e^{-2\beta J} \le \frac{1}{2} \tag{44}$$

Here, we have used the observation that a contour of length ℓ must be contained within a square box of size ℓ centered at x, along with lemma 2.2 and the fact that

$$\sum_{k=2}^{\infty} k^2 r^k = \frac{2r^2(2-\frac{3}{2}r+\frac{1}{2}r^2)}{(1-r)^3} \tag{45}$$

The lower bound on β now follows. □

Remark 2.2. *Clearly $\omega_\Lambda^+(\sigma_x)\to 1$ as $\beta\to\infty$, and with the same estimates for the $-$ boundary condition, we have that $\omega_\Lambda^-(\sigma_x)\to -1$ as $\beta\to\infty$. Hence, if*

$$\omega_\Lambda^\pm \to \omega^\pm \text{ as } \Lambda \nearrow \mathbb{Z}^2 \tag{46}$$

clearly, $\omega^+\neq\omega^-$. A similar argument works for $d\ge 2$, and these same arguments (using Peierls Estimate) can be generalized to other models with somewhat similar structure.

3 The Griffiths Inequalities

The goal for us is to show the existence of the limiting Gibbs states, but the Griffiths inequalities have many other applications. Again, we are considering

Ising systems on $\mathbb{Z}^d$. The algebra of observables for a finite volume $\Lambda \subset \mathbb{Z}^d$ is $C(\Omega_\Lambda)$. Consider the special observables

$$\sigma_A = \prod_{x\in A} \sigma_x \text{ for all } A \subseteq \Lambda \tag{47}$$

$$\sigma_\emptyset = 1 \tag{48}$$

We make the observation that the set $\{\sigma_A | A \subseteq \Lambda\}$ is a basis for $C(\Omega_\Lambda)$ because $\delta_{\eta_x=\varepsilon} = \frac{1}{2}(1+\varepsilon\sigma_x)$ forms a basis upon taking products. The ferromagnetic Ising model can be generalized to a general class of ferromagnetic models with local Hamiltonians of the form

$$H_\Lambda = -\sum_A J_A \sigma_A \ , \ J_A \geq 0 \tag{49}$$

Note that

$$\frac{\partial}{\partial J_B}\omega_\Lambda(\sigma_A) = \omega_\Lambda(\sigma_A\sigma_B) - \omega_\Lambda(\sigma_A)\omega_\Lambda(\sigma_B) \tag{50}$$

where we have set $\beta = 1$ in this equation.

Theorem 3.1 (Griffiths Inequalities)**.** *Let ω_Λ be the Gibbs state at β with ferromagnetic Hamiltonian H_Λ. Then*

1. $\omega_\Lambda(\sigma_A) \geq 0$ *for all* $A \subset \Lambda$.
2. $\omega_\Lambda(\sigma_A\sigma_B) - \omega_\Lambda(\sigma_A)\omega_\Lambda(\sigma_b) \geq 0$ *for all* $A, B \subset \Lambda$.

Proof. Proof of 1.

$$\omega_\Lambda(\sigma_A) = \frac{1}{Z_\Lambda}\sum_{\eta\in\Omega_\Lambda} \sigma_A(\eta)e^{-\beta H_\Lambda(\eta)} \tag{51}$$

$$= \frac{1}{Z_\Lambda}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\sum_{\eta\in\Omega_\Lambda}\sigma_A(\eta)(\sum_{B\subseteq\Lambda} J_B\sigma_B)^n \tag{52}$$

Clearly $\sigma_A\sigma_B = \sigma_C$ with $C = A\triangle B$, and where $A\triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B. Now, we group all terms with the same C.

$$\omega_\Lambda(\sigma_A) = \sum_C a(C)\sum_{\eta\in\Omega_\Lambda}\sigma_C(\eta) \tag{53}$$

If $C \neq \emptyset$, $\sum_{\eta\in\Omega_\Lambda}\sigma_C(\eta) = 0$ since if $x \in C$ then the sum over η with $\eta_x = \pm 1$ cancel each other out. In the case that $C = \emptyset$, $\sum_{\eta\in\Omega_\Lambda}\sigma_C = 2^{|\Lambda|}$. So $\omega_\Lambda(\sigma_A) = a(\emptyset)2^{|\Lambda|} \geq 0$.

Proof of 2.

Note that for 1, we did not really use the structure of $\mathbb{Z}^d$, and the argument works on any finite set Λ. Now we will consider $\widetilde{\Lambda} = \Lambda \sqcup \Lambda$, two disjoint copies of Λ. Equivalently, we can consider a system with two copies of the algebra

$$\widetilde{\mathcal{A}_\Lambda} = C(\Omega_\Lambda) \otimes C(\Omega_\Lambda) \tag{54}$$

where each copy of $C(\Omega_\Lambda)$ is generated by the functions σ_x and τ_x respectively. The configuration space is $\widetilde{\Omega_\Lambda} = \Omega_\Lambda \times \Omega_\Lambda = \{(\eta, \xi) | \eta_x, \xi_x \in \{-1, +1\}\}$. Similarly, we have σ_A and τ_A. Define

$$\widetilde{H_\Lambda}(\eta, \xi) = H_\Lambda(\eta) + H_\Lambda(\xi) \tag{55}$$

$$\widetilde{Z_\Lambda} = \sum_{\eta \in \Omega_\Lambda} \sum_{\xi \in \Omega_\Lambda} e^{-\beta H_\Lambda(\eta)} e^{-\beta H_\Lambda(\xi)} \tag{56}$$

$$= (Z_\Lambda)^2 \tag{57}$$

If $f(\eta, \xi) = f_1(\eta) f_2(\xi)$, we have

$$\widetilde{\omega_\Lambda}(f) = \omega_\Lambda(f_1)\omega_\Lambda(f_2) \tag{58}$$

Now consider "rotated" variables

$$s_x = \frac{1}{\sqrt{2}}(\sigma_x + \tau_x) \tag{59}$$

$$t_x = \frac{1}{\sqrt{2}}(\sigma_x - \tau_x) \tag{60}$$

which take values $-\sqrt{2}, 0, \sqrt{2}$ on double configurations. Note that

$$\sigma_x = \frac{1}{\sqrt{2}}(s_x + t_x) \tag{61}$$

$$\tau_x = \frac{1}{\sqrt{2}}(s_x - t_x) \tag{62}$$

and for $A \subseteq \Lambda$,

$$\Delta_A^{\pm} = \sigma_A \pm \tau_A = \left(\frac{1}{\sqrt{2}}\right)^{|A|} \{(s+t)_A \pm (s-t)_A\} \tag{63}$$

where $\sigma_A = \prod_{x \in A} \sigma_x$, and τ_A is completely analogous.

Lemma 3.1.

$$\Delta_A^{\pm} = \sum_{B \subseteq A} K_B s_{A \setminus B} t_B \tag{64}$$

with some $K_B \geq 0$

Proof. Just calculate

$$(s+t)_A = \prod_{x \in A} (s_x + t_x) \tag{65}$$

$$= \sum_{B \subseteq A} s_{A \setminus B} t_B \tag{66}$$

and

$$(s-t)_A = \sum_{B \subseteq A} (-1)^{|B|} s_{A \setminus B} t_B \tag{67}$$

The lemma follows. □

Now, we can proceed with the proof of 2.

$$\begin{aligned}
\omega_\Lambda(\sigma_A\sigma_B) - &\omega_\Lambda(\sigma_A)\omega_\Lambda(\sigma_B) && (68)\\
&= \widetilde{\omega_\Lambda}(\sigma_A\sigma_B) - \widetilde{\omega_\Lambda}(\sigma_A\tau_B) && (69)\\
&= \widetilde{\omega_\Lambda}(\sigma_A(\sigma_B - \tau_B)) && (70)\\
&= \left(\frac{1}{\sqrt{2}}\right)^{|A|+|B|} \widetilde{\omega_\Lambda}(s+t)_A\{(s+t)_B + (s-t)_B\} && (71)\\
&= \widetilde{\omega_\Lambda}(\sum_{C\subseteq B} K_C(s+t)_A s_{B\setminus C} t_C) && (72)\\
&= \widetilde{\omega_\Lambda}(\sum_D K_D s_{A\cup B\setminus D} t_D) && (73)
\end{aligned}$$

The variables s_x and t_x have the following properties:

$$s_x t_x = 0 \text{ and either } s_x \text{ or } t_x = 0 \text{ for a given configuration } \sigma_x, \tau_x \qquad (74)$$

t_x and s_x are odd functions of σ_x and τ_x, so all their odd powers are odd and all their even porwers are of course greater than or equal to 0. So

$$\sum_{\sigma_x,\tau_x} s_x^n = \begin{cases} = 0 & \text{if } n \text{ odd.} \\ > 0 & \text{if } n \text{ even.} \end{cases} \qquad (75)$$

and the same result holds for $\sum t_x^n$. The Hamiltonian can be rewritten,

$$\begin{aligned}
\widetilde{H_\Lambda} &= \sum_{A\subset\Lambda} K_A\sigma_A + K_A\tau_A && (76)\\
&= \sum_{A\subset\Lambda} \widetilde{K_A} \sum_{C\subset A} (1+(-1)^{|C|}) s_{A\setminus C} t_C && (77)
\end{aligned}$$

and is again a Hamiltonian with coefficients greater than or equal to 0 in the monomial basis. Although the polynomials in s and t are not independent variables, the same argument as in 1 applies.

$$\begin{aligned}
\widetilde{H_\Lambda} &= \sum_{C,D\subset\Lambda} \widetilde{K_{C,D}} s_C t_D && (78)\\
\widetilde{\omega_\Lambda} &= \frac{1}{\widetilde{Z_\Lambda}} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \sum_{\{\sigma_x,\tau_x\}} s_A t_B (\sum_{C,D\subset A} \widetilde{K_{C,D}} s_C t_D)^n && (79)\\
&= \frac{1}{\widetilde{Z_\Lambda}} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \sum_{\{\sigma_x,\tau_x\}} \sum_{C,D} \widetilde{\widetilde{K_{ABCD}}} s_A s_C t_B t_D && (80)
\end{aligned}$$

Indeed,

$$\begin{aligned}
\widetilde{H_\Lambda} &= \sum_{A\subset\Lambda} J_A\sigma_A + J_A\tau_A && (81)\\
&= \frac{1}{\sqrt{2}} \sum_{A\subset\Lambda} \sum_{C\subset A} J_A(1+(-1)^{|C|}) s_{A\setminus C} t_C && (82)
\end{aligned}$$

Therefore we can apply 1 to $\widetilde{H_\Lambda}$ and finish the proof of 2. □

Recall the observation

$$\frac{\partial}{\partial J_B}\omega_\Lambda(\sigma_A) = \omega_\Lambda(\sigma_A\sigma_B) - \omega_\Lambda(\sigma_A)\omega_\Lambda(\sigma_B) \tag{83}$$

therefore the second Griffiths inequality implies that $\frac{\partial}{\partial J_B}\omega_\Lambda(\sigma_A) \geq 0$ for ferromagnetic Ising models.

4 The Thermodynamic Limit of Ising Equilibrium States

The set of states on $C(\Omega_{\mathbb{Z}^d})$ is weak-* compact. From this we deduce that at each fixed βJ, the set of finite volume states $\{\omega_\Lambda\}_{\Lambda\subseteq\mathbb{Z}^d}$ has at least one limit point (extend them to states on all of $\mathbb{Z}^d$ in more or less any way you like). And more generally the same is true for sequences with other boundary conditions b_Λ. The Peierls argument shows that if ω^+ and ω^- are such limit points of $\{\omega_\Lambda^+\}$ and $\{\omega_\Lambda^-\}$ respectively, then for all β large enough, they will be distinct, since

$$\omega^+(\sigma_0) = -\omega^-(\sigma_0) \neq 0 \tag{84}$$

Later, in a more general context, we will show that for small β the limit points are unique independent of b_Λ. It is nevertheless still an interesting question whether the sequence ω_Λ^+ itself converges.

Theorem 4.1. *1. Let $\{\omega_\Lambda^0\}$ be the sequence of β Gibbs states in finite volume $\Lambda \subseteq \mathbb{Z}^d$ of the Ising model with free boundary conditions. Then*

$$\omega_\Lambda^0(\sigma_A) \nearrow \omega^0(\sigma_A) \text{ for all finite subsets } A \subset \mathbb{Z}^d \tag{85}$$

2. If $\{\omega_\Lambda^+\}$ is a sequence corresponding to + boundary conditions, then

$$\omega_\Lambda^+(\sigma_A) \searrow \omega^+(\sigma_A) \tag{86}$$

i.e., we have weak convergence in both cases and they are monotone increasing and decreasing, respectively, on the basis functions σ_A.*

Proof. Proof of 1.

ω_Λ^0 can be regarded as the Gibbs state for the ferromagnetic Ising model

$$H_\Lambda = -\sum_X J_X^\Lambda \sigma_X \tag{87}$$

with

$$J_X^\Lambda = \begin{cases} J & \text{if } X=(x,y), x,y\in\Lambda, |x-y|=1 \\ 0 & \text{otherwise} \end{cases} \tag{88}$$

with $J > 0$. Hence, J_X^Λ is monotonic increasing by the second Griffiths inequality.

Proof of 2.

Define

$$J_X^\Lambda = \begin{cases} J_x & \text{if } X = x, y, x, y \in \Lambda, |x - y| = 1 \\ +\infty & \text{if } X = \{x\}, x \in \Lambda^c \\ 0 & \text{otherwise} \end{cases} \tag{89}$$

It is not hard to see that the infinite coupling constant does not pose a problem.

□

Note that the Griffiths inequalities also show that a variety of other Ising models with higher dimensionality and anisotropies also have a non-vanishing magnetism at sufficiently low temperature by comparing with the two-dimensional translation invariant model.