On the distribution of eigenvalues and their spacings: a rapid introduction to random matrix theory

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Abstract

This is intended to provide a brief intro to the main issues and tools of random matrix theory. The main problem we will be discussing will be the determination of a probability distribution for the eigenvalues of certain families of matrices. We will also be interested in the distribution of the spacings between the eigenvalues. Random matrix theory was born from the adhoc but reasonable assumption that the behavior of energy levels inside any sufficiently complicated physical system can be simulated by that of the eigenvalues of a suitably chosen family of random matrices. The physical considerations (symmetry, time reversal invariance, etc ...) dictate the choice of the particular form of the matrices considered. This statistical line of thinking is originally the applied mathematicians' way of tackling some otherwise computationally intractable problems of Quantum Physics. However, Random Matrix theory has since then found applications to many branches of pure mathematics itself. To illustrate but one of those many connections, assuming the Riemann Hypothesis to be true, it was shown that the zeros of $\zeta(z)$ behave like the eigenvalues of the matrices chosen from a particular family (the Gaussian Unitary Ensemble; see definition below).

1 Distribution of Eigenvalues

In Quantum Mechanics, properties of a system in a given state are represented by eigenvalues of a symmetric or Hermitian linear operator. For instance the energy levels are the solutions of the eigenvalue equation $H\psi = E\psi$. If the system is complicated enough, these matrices are much too large for any practical computational approach even if they are known to be finite. It is practically impossible to compute the entries of H. Therefore it is reasonable to consider H as a random matrix as $N \to \infty$. In classical Statistical Mechanics for example positions and momenta of particles are treated as random variables since one is interested in calculating average properties such as collision cross sections or total energy. Similarly in quantum Statistical Mechanics, when average properties of the system are of interest, one can make a similar assumption, namely that the linear operators are random matrices. Hence the behavior of

the average properties of the system is simulated by the behavior of the eigenvalues of random matrices. Thus we want to study the behavior (distribution) of the eigenvalues of some subset of all $N \times N$ matrices subject to some probability distribution $P(H)$ as $N \to \infty$. It turns out that the specific physical properties of the system at hand dictate the form of the matrices to be studied (the subset) and the probability distribution P they obey (c.f. [1], chapter 2). There are three main cases of interest:

- systems that are invariant under time–reversal and have even-spin. This implies that H is symmetric Hermitian, its upper–half entries $H_{ij}, i \geq j$ are statistically independent, and it is invariant real orthogonal transformations. Mathematically this translates into the following:
  1. the diagonal and upper–half elements $H_{ij}, i \geq j$ are i.i.d. random variables, so that $P(H) = \sum_{i \geq j} f_{ij}(H_{ij})$, where f_{ij} is the probability distribution for H_{ij}.
  2. the probabilty $P(H)dH = \sum_{i \leq j} f_{ij}(H_{ij})$ that the system will belong to the volume element dH, is invariant under all real orthogonal transformations, so that $P(O^T HO)dH = P(H)dH$ for all O real such that $O^{-1} = O^T$.

  The subset of all $N \times N$ matrices that satisfies the above conditions is called the Gaussian Orthogonal Ensemble (GOE). The origin of the "Gaussian" part will be obvious soon.
- systems that are invariant under time–reversal and have odd-spin. This implies the following: H is Hermitian; its diagonal entries H_{ii} and the four quaternionic components of its upper–half entries $H_{ij}, i > j$ are statistically independent; H is invariant under all symplectic transformations. Mathematically this translates into:
  1. the p.d.f. $P(H)$ is a product of functions of one variable: $P(H) = \prod_{i \leq j} f_{ij}^{(0)}(H_{ij}^{(0)}) \prod_{\lambda=1}^{3} \prod_{i<j} f_{ij}^{\lambda}(H_{ij}^{(\lambda)})$.
  2. the probabilty $P(H)dH$ that the system will belong to the volume element $dH = \prod_{i \leq j} dH_{ij}^{(0)} \prod_{\lambda=1}^{3} \prod_{i<j} dH_{ij}^{(\lambda)}$, is invariant under all symplectic transformations, so that $P(S^{-1}HS)dH = P(H)dH$ for all S symplectic.

  The subset of all $N \times N$ matrices that satisfies the above conditions is called the Gaussian Symplectic Ensemble (GSE).
- systems that are *not* time–reversal invariant. This implies the following: H is Hermitian; its diagonal entries H_{ii} and the real and imaginary parts of its upper–half entries $H_{ij}, i > j$ are statistically independent; H is invariant under all unitary transformations. Mathematically this translates into:
  1. the p.d.f. $P(H)$ is a product of functions of one variable: $P(H) = \prod_{i \leq j} f_{ij}^{(0)}(H_{ij}^{(0)}) \prod_{i<j} f_{ij}^{(1)}(H_{ij}^{(1)})$ where the subscripts $0, 1$ denote the real and imaginary parts.
  2. the probabilty $P(H)dH$ that the system will belong to the volume element $dH = \prod_{i \leq j} dH_{ij}^{(0)} \prod_{i<j} dH_{ij}^{(1)}$, is invariant under all unitary transformations, so that $P(U^{-1}HU)dH = P(H)dH$ for all U unitary.

The subset of all $N \times N$ matrices that satisfies the above conditions is called the Gaussian Unitary Ensemble (GUE).

One can prove that for these three ensemple, P must be a distribution of the form $P(H) = e^{-a \operatorname{tr} H^2 + b \operatorname{tr} H + c}$. This reminds the usual gaussian distribution for real random variable, hence the name on the ensembles. We shall prove this result for the GOE. The other cases are similar. We shall first need two lemmas:

Lemma 1.1 (Weyl, 1946). *Let H be $N \times N$ symmetric and $f(H)$ be invariant under nonsingular similarity transformations, so that $f(h') = f(h)$ is $h' = AHA^{-1}$. Then:*

$$f = f(\operatorname{tr} H, \operatorname{tr} H^2, \ldots, \operatorname{tr} H^N)$$

That is, f is a function of the traces of the first N powers of H.

Proof. f is a symmetric function of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. A basic fact of the theory of symmetric functions is that they admit the so–called power symmetric functions

$$p_j = \sum_{k=1}^{N} \lambda_k^j \qquad j = 1, \ldots, N$$

as a basis. But p_j is exactly the trace of H^j.

□

Lemma 1.2. *If three differentiable functions $f_k(x), k = 1, 2, 3$ satisfy the equation:*

$$f_1(xy) = f_2(x) + f_3(y) \tag{1}$$

Then they are of the form $f(x) = a \ln(x) + b_k, k = 1, 2, 3$ with $b_1 = b_2 + b_3$.

Proof. Differentiate 1 with respect to x to get:

$$f_1'(xy) = \frac{1}{y} f_2'(x) \tag{2}$$

which, integrated with respect to y gives:

$$\frac{1}{x} f_1(xy) = f_2'(x) \ln y + \frac{1}{x} g(x) \tag{3}$$

for an arbitrary $g(x)$. Substitute for $f_1(xy)$ from 3 into 1 and get:

$$x f_2'(x) \ln y + g(x) - f_2(x) = f_3(y) \tag{4}$$

The right–hand side depends only on y,so the left–hand side must be independent of x which implies:

$$x f_2'(x) = a \qquad g(x) - f_2(x) = b_3 \tag{5}$$

so that :

$$f_2(x) = a \ln x + b_2 = g(x) - b_3 \tag{6}$$

for arbitrary a, b_2, b_3. But we also get from 4:

$$f_(3)(y) = a \ln y + b_3 \tag{7}$$

and 1 gives:

$$f_1(xy) = a \ln(xy) + (b_2 + b_3) \tag{8}$$

□

Theorem 1.3. *Suppose a probability density $P(H)$ on $N \times N$ random Hermitian symmetric matrices H satifies the following conditions:*

- *the upper–half elements $H_{ij}, i \geq j$ are i.i.d. random variables, so that $P(H) = \prod_{i \geq j} f_{ij}(H_{ij})$, where f_{ij} is the probability distribution for H_{ij}.*
- *P is invariant under real orthogonal transformations; that is, for O real such that $O^{-1} = O^T$, $P(O^T H O) = P(H)$.*

Then P is of the form

$$P(H) = e^{-a \operatorname{tr} H^2 + b \operatorname{tr} H + c}$$

Proof. Consider the orthogonal transformation that consist of a two–dimensional rotation by θ:

$$O = \begin{pmatrix} \cos\theta & \sin\theta & 0 & \dots & 0 \\ -\sin\theta & \cos\theta & 0 & \dots & 0 \\ 0 & 0 & & 1 & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \dots & & 1 \end{pmatrix}$$

Let $H' = O^T H O \Rightarrow H = O H' O^T$. Therefore:

$$\begin{aligned} \frac{\partial H}{\partial \theta} &= \frac{\partial}{\partial \theta}(O H' O^T) \\ &= \frac{\partial O}{\partial \theta}(H' O^T) + O H' \frac{\partial O^T}{\partial \theta} \\ &= \frac{\partial O}{\partial \theta}(O^T H O O^T) + O O^T H O \frac{\partial O^T}{\partial \theta} \\ &= \frac{\partial O}{\partial \theta} O^T H + H O \frac{\partial O^T}{\partial \theta} \\ &= AH + HA^T \qquad \text{for} \qquad A = \frac{\partial O}{\partial \theta} O^T \end{aligned}$$

Now $P(H) = \prod_{i \geq j} f_{ij}(H_{ij})$ so the logarithmic derivative of P is:

$$\frac{\partial P}{\partial \theta} = \frac{\partial\, e^{\ln P}}{\partial \theta} = \frac{\partial \ln P}{\partial \theta} e^{\ln P} = \frac{\partial}{\partial \theta} \left(\sum \ln(f_{ij}(H_{ij})) \right) e^{\ln P}$$

By assumption P is invariant under this orthogonal transformation so:

$$\begin{aligned} \frac{\partial P}{\partial \theta} = 0 \quad &\Rightarrow \quad \frac{\partial}{\partial \theta} \left(\sum \ln(f_{if}(H_{ij})) \right) = 0 \\ &\Rightarrow \quad \sum \frac{1}{f_{ij}} \frac{\partial f_{ij}}{\partial H_{ij}} \frac{\partial H_{ij}}{\partial \theta} = 0 \end{aligned}$$

Substituting in for $\frac{\partial H_{ij}}{\partial \theta}$ gives:

$$\left(\frac{1}{f_{11}}\frac{\partial f_{11}}{\partial H_{11}} + \frac{1}{f_{22}}\frac{\partial f_{22}}{\partial H_{22}}\right)(2H_{12}) + \left(\frac{1}{f_{12}}\frac{\partial f_{12}}{\partial H_{12}}\right)(H_{11}+H_{22}) + \sum_{k=3}^{N}\left(\frac{1}{f_{1k}}\frac{\partial f_{1k}}{\partial H_{1k}} + \frac{1}{f_{2k}}\frac{\partial f_{2k}}{\partial H_{2k}}\right) = 0$$

But each term depends on an independent random variable, so they each must be contants. Hence, dividing by $H_{1k}H_{2k}$:

$$-\frac{1}{H_{1k}f_{1k}}\frac{\partial f_{1k}}{\partial H_{1k}} + \frac{1}{H_{2k}f_{2k}}\frac{\partial f_{2k}}{\partial H_{2k}} = \frac{C_k}{H_{1k}H_{2k}}$$

This is an equatio of the form $f_1(xy) = f_2(x) + f_3(y)$, which by the lemma, has solutions of the form $a + b\ln x$. So $C_k = 0$ and :

$$\frac{1}{f_{1k}}\frac{\partial f_{1k}}{\partial H_{1k}} = \frac{1}{H_{2k}f_{2k}}\frac{\partial f_{2k}}{\partial H_{2k}} = c$$

Integrating gives:

$$f_{1k} = e^{\frac{a}{2}X_{1k}^2}$$

We get the same for f_{ij} whenever $i \neq j$. Now $P(H)$ is an invariant under a similarity transformation of symmetric matrices so by the above lemma it must be a function of the traces of the first N powers of H. The off-diagonal terms as we just saw depend only on $\operatorname{tr} H^2$. Hence $P(H)$ is an exponential that depends at most on $\operatorname{tr} H$ and $\operatorname{tr} H^2$. Hence P is of the form:

$$e^{-a\operatorname{tr} H^2 + b\operatorname{tr} H + c}$$

□

Now that we have shown what the distribution function must look like, given certain physical considerations, we are ready to give the probability distribution for the eigenvalues. Here again we will restrict ourselves to the case of the Gaussian Orthogonal Ensemble, but the proofs are similar in the other two cases.

Theorem 1.4 (Eigenvalue Distribution for symmetric matrices). *Let H be an $N \times N$ symmetric matrix drawn from a probability density $\mu(\lambda_1, \ldots, \lambda_N)$ with repesct to the Lebesgue measure $\prod_{i\leq j} dH_{ij}$ on $\mathbb{R}^{\frac{n(n-1)}{2}}$ where μ is a function of the eigenvalues of H arranged in increasing order. Then the joint probability distribution function $P_N(\lambda_1, \ldots, \lambda_N)$ for the eigenvalues on $\mathbb{R}^n_{\lambda_1<\cdots<\lambda_N}$ is:*

$$P_N(\lambda_1, \ldots, \lambda_N) = K_N \mu(\lambda_1, \ldots, \lambda_N) \prod_{i<j} |\lambda_i - \lambda_j|$$

Proof. Let us construct the measure on the eigenvalue space corresponding to the standard Lebesgue measure. With probability 1, $H = ODO^T$

where D is a diagonal matrix with distinct eigenvalues, and O is orthogonal. Hence $OO^T = I$ and $dO^T \cdot O = -O^T \cdot dO$. Differentiate H and substitute in $dM = dO^T \cdot O$ to get:

$$dH = dO \cdot D \cdot O^T + O \cdot dD \cdot O^T + O \cdot D \cdot dO^T$$

Now:

$$\begin{aligned} dO \cdot D \cdot O^T &= O \cdot O^T \cdot dO \cdot D \cdot O^T \\ &= -O \cdot dO^T \cdot O \cdot D \cdot O^T \\ &= -O \cdot dM \cdot D \cdot O^T \end{aligned}$$

And:

$$\begin{aligned} O \cdot D \cdot dO^T &= O \cdot D \cdot dO^T \cdot O \cdot O^T \\ &= O \cdot D \cdot dM \cdot O^T \end{aligned}$$

Hence:

$$dH = O \cdot (dD - dM \cdot D + D \cdot dM) \cdot O^T$$

Since the measure dH is invariant under orthogonal transformations, we get:

$$\begin{aligned} \prod_{i \le j} dH_{ij} &= \prod_{i \le j} (dD - dM \cdot D + D \cdot dM)_{ij} \\ &= \prod_{i<j} |\lambda_i - \lambda_j| dM_{ij} \prod_{i=1}^{N} \lambda_i \\ &= \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} d\lambda_i \prod_{i<j} dM_{ij} \end{aligned}$$

Integrating out the dM_{ij} we find that the induced measure is:

$$K_N \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} d\lambda_i$$

Hence if H has density $\mu(\lambda_1, \ldots, \lambda_N) \prod_{i \le j} dH_{ij}$ then the probability density in the eigenvalue space is:

$$K_N \mu(\lambda_1, \ldots, \lambda_N) \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} d\lambda_i$$

□

Corollary 1.5. *The probability that a GOE matrix H has eigenvalues $E_1, \ldots, E_N$ is:*

$$P_N(\lambda_1, \ldots, \lambda_N) = K_N . \prod_{i<j} |\lambda_i - \lambda_j| \cdot \exp\left(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\right)$$

Proof. This is a straightforward application of the last two theorems. □

REMARK If we shift the origin of the λ_i to $b/2a$ and change the energy scale everywhere by a constant factor $\sqrt{2a}$ we may replace λ_i by $(1/2a)x_i + b/2a$. Hence P_N takes the simpler form:

$$P_N(x_1, \ldots, x_N) = C_N . \prod_{i<j} |x_i - x_j| \cdot \exp\left(-\frac{1}{2} \sum x_i^2\right)$$

$P_N(E_1, \ldots, E_N)$ gives the probability of finding the N eigenvalues of H in unit intervals around $E_1, \ldots, E_N$. Recall that if we want to study the average behavior of the system, we would like to know the probability $\sigma(x) = P_\infty(x)$ that H has an eigenvalue in a unit iterval around x inthe limit $N \to \infty$. Technically, this could be achieved in the following manner: we can get the probability $P_N(x)$ that H will have an eigenvalue around in a unit inteval around x by integrating out $N-1$ of the variables in the above equation:

$$P_N(x) = N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x, E_2 \ldots, E_N)\, dE_2 \cdots dE_N$$

Next just take $\lim_{N\to\infty} P_N(x)$. Mehta and Gaudin [2] actually carried out the process this way. However the calculations are pretty tedious. We shall instead derive this in a different fashion, thereby showing some interesting connections to potential theory, in particular the theory of equilibrium measures, and orthogonal polynomials. Using the above theorem, one rigorously derive the celebrated Wigner semicircle law:

Theorem 1.6 (Wigner Semicircle law). *As $N \to \infty$ the distribution of the normalized eigenvalues tends to the semicircle distributions. That is:*

$$P_N(x) \xrightarrow{N\to\infty} \frac{1}{\pi}\sqrt{2 - x^2}$$

Proof. Surprise! □

References

[1] M. L. Mehta. *Random Matrices, Revised and Enlarged Second Edition.* Academic Press, 1991.

[2] M. L. Mehta and M. Gaudin. On the density of eigenvalues of a random matrix. *Nuclear Physics*, 18:420, 1960.