The Mermin-Wagner Theorem

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1 The main energy-entropy balance argument

Let $\mathcal{A}$ be a C^* algebra such as the algebra of quasi-local observables of a quantum spin system on $\mathbb{Z}^d$, and suppose $\{\alpha_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of automorphisms of $\mathcal{A}$, which we will refer to as the *dynamics* of the system. The examples we have in mind are the dynamics of a quantum spin system generated by a not-too-long-range interaction Φ, e.g., one that satsifies, for some $\lambda > 0$,

$$\|\Phi\|_\lambda \equiv \sup_{x\in\mathbb{Z}^d} \sum_{x\ni X} e^{\lambda|X|} \|\Phi(X)\| \, < \, \infty \, .$$

A *symmetry* of the system is an automorphism, τ, of $\mathcal{A}$, which commutes with α_t, i.e.,

$$\alpha_t(\tau(A)) = \tau(\alpha_t(A)), \quad \text{for all } A \in \mathcal{A}, t \in \mathbb{R}$$

It is easy to see that if τ is a symmetry, than so is τ^{-1}. In fact, the set of all automorphisms commuting with the dynamics is a group for composition of automorphisms.

It is easy to see that if τ is a symmetry and ω is a $\beta-$KMS state for α_t, then $\omega \circ \tau$ is also $\beta-$KMS. The Mermin-Wagner-Hohenberg Theorem gives sufficient conditions that imply that all $\beta-$KMS states, ω, of the system are τ-invariant, i.e., $\omega(\tau(A)) = \omega(A)$, for all $A \in \mathcal{A}$. The original theorem, a special case of what we will prove here, says that no spontaneaous breaking of any continuous symmetry occurs at finite temperatures ($\beta < \infty$) in dimensions $d \leq 2$.

The general theorem involves the following two assumptions, which we will verify for a variety of systems, including two-dimensional models with a continuous symmetry.

MWH1: The symmetry τ is approximately inner in the sense that there exist a sequence of unitaries $U_n \in \mathcal{A}$ such that

$$\lim_{n\to\infty} \|\tau(A) - U_n^* A U_n\| = 0, \quad \text{for all } A \in \mathcal{A} \, .$$

We also assume that these unitaries can be taken form the domain of δ, the generator of the dynamics $\alpha_t = e^{it\delta}$. Equivalently, we assume that the follwing limits exist

$$\lim_{t\to 0} \frac{\alpha_t(U_n) - U_n}{t} = i\delta(U_n) \, .$$

Note that it follows from these assumptions that τ^{-1} is also approximately inner, approximated by the unitaries U_n^*, and that $U^* \in \mathrm{Dom}(\delta)$.

The second assumption comes in two versions.

MWH2: We assume that one of the following holds:
(i) there exists a constant M such that $\|\delta(U_n)\| \leq M$, for all n, or
(ii) all $\beta-$KMS states are τ^2-invariant and there exists a constant M such that

$$\|U_n^*\delta(U_n) + U_n\delta(U_n^*)\| \leq M, \quad \text{for all } n\,.$$

Theorem 1.1 *Suppose τ is a symmetry of the system $(\mathcal{A}, \alpha_t)$ such that conditions MWH1 and MWH2 ((i) or (ii)) are satisfied. Then, all $\beta-$KMS states are $\tau-$ invariant for all $\beta \in [0, \infty)$.*

Using the assumptions and the EEB inequalities, we will prove that if ω is $\beta-$KMS, then there exists a constant C such that

$$\omega \circ \tau(A^*A) \leq C\omega(A^*A) \tag{1.1}$$

The constant C will depend only on β and M. This a uniform version of absolute continuity of $\omega\circ\tau$ with respect to ω. It is not hard to prove that for extremal $\beta-$KMS states one has the dichotomy: either they are equal or they are disjoint. That is, if they are quasi-equivalent states, *a fortiori*, if one is absolutely continuous with respect to the other, then they are necessarily equal. This follows from the general result that (1.1) implies that there exists $0 \leq T \in \pi_\omega(\mathcal{A})' \cap \pi_\omega(\mathcal{A})''$ such that $\omega \circ \tau(A^*A) = \langle \Omega_\omega, \pi_\omega(A^*A)T\Omega_\omega\rangle$. Since extremal KMS states are factor states, such T must be a multiple of $\mathbb{1}$ and, therefore, $\omega \circ \tau = \omega$.

So, from (1.1), it will follow that all extremal $\beta-$KMS states are $\tau-$invariant and, therefore, by taking convex combinations, all $\beta-$KMS states are $\tau-$invariant.

The second version of MWH2 includes the assumption that we already know that the $\beta-$KMS states are τ^2-invariant. This is no restriction for compact continuous summetry groups. For discrete groups such as finite groups or lattice translations one needs version (i). In general, (i) implies (ii), but note that for involutions ($\tau^2 = \mathbb{1}$), (i) and (ii) are equivalent. Now, we prove Theorem refthm:main.

Proof: Let ω be a $\beta-$KMS state. To prove (1.1) we will use the EEB inequalities and the GNS reprensentation of ω. As any KMS state is time invariant, α_t is unitarily implemented by unitaries U_t in the GNS representation. As α_t is strongly continuous, U_t is a strongly continuous one-parameter group generated by a s.a. operator H, with dense domain $\mathrm{Dom}(H)$, and such that $H\Omega = 0$, where Ω is the cyclic vector representing ω. We will need the spectral resolution of H:

$$H = \int \lambda \, dE_\lambda$$

to define a resolution of the identity by mutually orthogonal projections $P_n, n \in \mathbb{Z}$, $\sum_n P_n = \mathbb{1}$, as follows

$$P_n = \int_{(n\epsilon, n+1]} dE_\Lambda$$

It is clear that, to prove (1.1), it is sufficient to prove that there exists a constant C, independent of n, such that for all $A \in \mathcal{A}$

$$\omega \circ \tau(A^*P_nA) \leq C\omega(A^*P_nA)$$

or more accurately, we will prove that, for all m, n, and for $A \in \mathcal{A}_0$, a norm-dense $*-$subalgebra of $\mathcal{A}$, we have

$$\langle \Omega \mid U_m^* \pi(A^*) P_n \pi(A) U_m \Omega \rangle \leq C \langle \Omega \mid \pi(A^*) P_n \pi(A) \Omega \rangle \tag{1.2}$$

By summing over n and taking the limit $m \to \infty$ one obtains (1.1).

To prove (1.2) we need the following estimates for quantities that appear in the EEB inequalities. For convenience, we introduce the notation $A_n = P_n \pi(A)$. For the first estimate, note that vectors of the form $A_n \Omega$ are in the domain of H. We will also use $\omega(\,\cdot\,)$ as shorthand for $\langle \Omega \mid \cdot\, \Omega \rangle$. Then we have, by using $H\Omega = 0$,

$$\begin{aligned}
\omega(A_n^* \delta(A_n)) &= \langle \Omega \mid A_n^* P_n H P_n A_n \Omega \rangle \\
&\leq (n+1)\,\langle \Omega \mid A_n^* P_n H P_n A_n \Omega \rangle \\
&\leq (n+1)\,\omega(A_n^* A_n)
\end{aligned}$$

For the entropy term, we first observe that, using the KMS condition, we can relate $\omega(A_n^* A_n)$ and $\omega(A_n A_n^*)$ as follows:

$$\begin{aligned}
\omega(A_n A_n^*) &= \omega(A_n^* \alpha_{i\beta}(A_n)) \\
&= \langle \Omega \mid A_n^* P_n E^{-\beta H} P_n A_n \Omega \rangle \\
&\leq e^{-\beta n} \omega(A_n^* A_n)
\end{aligned}$$

From this estimate we get

$$\begin{aligned}
\omega(A_n^* A_n) \log \frac{\omega(A_n^* A_n)}{\omega(A_n A_n^*)} &\geq \omega(A_n^* A_n) \log \frac{\omega(A_n^* A_n)}{e^{-\beta n \epsilon} \omega(A_n^* A_n)} \\
&\geq \beta n \epsilon\, \omega(A_n^* A_n)
\end{aligned}$$

The EEB inequality for the observable $X = U_m A_n$:

$$\beta \omega(A_n^* U_m^* \delta(U_m A_n)) \geq \omega(A_n^* A_n) \log \frac{\omega(A_n^* A_n)}{\omega(U_m A_n A_n^* U_m^*)}$$

By using the derivation property on the left and adding and substracting a term on the right, and reorganizing this can be written as (watch the stars!)

$$\omega(A_n^* A_n) \log \frac{\omega(A_n A_n^*)}{\omega(U_m A_n A_n^* U_m^*)} \leq \beta \omega(A_n^* U_m^* \delta(U_m) A_n) + \beta \omega(A_n^* \delta(A_n)) - \omega(A_n^* A_n) \log \frac{\omega(A_n^* A_n)}{\omega(A_n A_n^*)}$$

The last two terms can be bounded by the estimates we prepared. The result gives

$$\omega(A_n^* A_n) \log \frac{\omega(A_n A_n^*)}{\omega(U_m A_n A_n^* U_m^*)} \leq \beta \omega(A_n^* U_m^* \delta(U_m) A_n) + \beta \omega(A_n^* A_n) \tag{1.3}$$

Now it is time to use MWH2. The two versions are treated slightly differently. With version (i), we immediately get

$$\omega(A_n^* A_n) \log \frac{\omega(A_n A_n^*)}{\omega(U_m A_n A_n^* U_m^*)} \leq \beta(M+1) \omega(A_n^* A_n).$$

After simplifying, exponentiating, and reversing the roles of A_n and A_n^*, as well as τ and τ^{-1}, one gets (1.1) with $C = e^{\beta(M+1)}$.

In order to use MWH2 (ii), we use (1.3) and the similar bound for U_m and U_m^* interchanged. By adding the two bounds we get:

$$\begin{aligned} &\omega(A_n^* A_n) \log \frac{\omega(A_n A_n^*)^2}{\omega(U_m A_n A_n^* U_m^*)\omega(U_m^* A_n A_n^* U_m)} \\ \leq\ & \beta\omega(A_n^*[U_m^*\delta(U_m) + U_m\delta(U_m^*)]A_n) + 2\beta\,\omega(A_n^* A_n) \end{aligned}$$

In the same way as before, but by using (ii) instead of (i), we obtain

$$\omega(A_n A_n^*)^2 \leq e^{\beta(M+2)}\omega(\tau(A_n A_n^*)\omega(\tau^{-1}(A_n A_n^*))$$

As $\omega \circ \tau$ is a $\beta-$KMS state, too, we can write

$$\omega(\tau(A_n A_n^*))^2 \leq e^{\beta(M+2)}\omega(\tau^2(A_n A_n^*)\omega(A_n A_n^*)$$

Now, we have to used that $\beta-$KMS states are τ^2-invariant. By taking square roots we get (1.1) with $C = e^{\beta(M+2)/2}$. ■

2 Applications. The Mermin-Wagner-Hohenberg Theorem

Recall that the assumption MWH2 of Theorem 1.1 came in two versions. We assumed that one of the following holds: (i) there exists a constant M such that $\|\delta(U_n)\| \leq M$, for all n; (ii) all $\beta-$KMS states are τ^2-invariant and there exists a constant M such that

$$\|U_n^*\delta(U_n) + U_n\delta(U_n^*)\| \leq M, \quad \text{for all } n\,.$$

We still need to show how the theorem is used to prove absence of continuous symmetry breaking in two dimensions at finite temperature. As we will show a bit further, MWH2 (ii), but not (i), can be verified in this case. The following lemma allows us to apply the main theorem to continuous symmetries.

Lemma 2.1 *Let $\{tau_\phi \mid \phi \in S^1\}$ be a compact connected continuous one-parameter group of automorphisms of $\mathcal{A}$. Let K be a set of states ω such that $\omega \circ \tau_\phi^2 = \omega$ implies $\omega \circ \tau_\phi = \omega$, for any $\phi \in S^1$. Then all $\omega \in K$ are $\tau_\phi-$invariant for all $\phi \in S^1$.*

Proof: As $\tau_\pi^2 = \text{id}$, the assumptions imply that $\omega \circ \tau_\pi = \omega$. By repeating the argument n more times we get that $\omega \circ \tau_{\pi/2^n}$. It follows immediately that ω is invariant for all τ_ϕ with ϕ of the form $\phi = \sum_{n=0}^N a_n 2^{-n}\pi$, where $a_n \in \mathbb{Z}$. Clealry, such ϕ form a dense set in S^1. Now, for every $A \in \mathcal{A}$, the function $\phi \to \omega(\tau_\phi(A) - A)$ is continous and vanishes on dense subset of S^1. Hence, it vanishes everywhere. ■

For symmetries representing an arbitrary compact Lie group, we can apply this lemma for a generating set of one-dimensional compact subgroups.

Now, we will verify MWH1 and MWH2 (ii) for two-dimensional quantum spin systems with a connected compact continuous symmetry group. For simplicity we will consider pair interactions only. This means that the dynamics is generated by local Hamiltonians of the form

$$H_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\Phi_{x,y} \tag{2.1}$$

for finite subsets Λ in $\mathbb{Z}^2$. where $\Phi_{x,y} \in \mathcal{A}_{\{x,y\}}$ are assumed to be uniformly bounded: say $\|\Phi_{x,y}\| \leq 1$, for all $x, y \in \mathbb{Z}^2$. Boundary terms are irrelevant in our considerations here. Suppose that there there are unitary representations

$$U_x(\phi) = e^{i\phi X_x}, \phi \in S^1,$$

with generators $X_x = X_x^* \in \mathcal{A}_{\{x\}}$, $x \in \mathbb{Z}^2$. E.g., for spin rotations the generators are SU(2) spin matrices.

Consider the boxes $\Lambda_m = [-m, m]^2 \subset \mathbb{Z}^2$. It is easy to satisfy MWH1 with a sequence of unitaries of the form

$$U_m(\phi) = \bigotimes_{x\in\Lambda_{2m}} U_x(\phi_m(x))$$

where $\phi_m(x) = \phi$, for all $x \in \Lambda_m$ and, for the moment, arbitrary for $x \in \Lambda_2 m \setminus \Lambda_m$.

Translation invariance is not required; in fact the argument works for inhomogeneous systems with spins of different magnitudes at different sites. We will assume that there is a uniform bound on the norm of the generators, say, there is a constant G such that $\|X_x\| \leq G$, for all $x \in \mathbb{Z}^2$.

Proposition 2.2 *For a quantum spin system on $\mathbb{Z}^2$ with local Hamiltonians of the form (2.1), with coupling constants $J(x, y)$, satisfying*

$$\sup_x \sum_{y\in\mathbb{Z}^2} |x-y|^2 |J(x,y)| < +\infty$$

we can find $\phi_m(x)$ such that there exists a consant M such that

$$\|U_m\delta(U_m^*) + U_m^*\delta(U_m)\| \leq M, \text{ for all } m$$

and Theorem 1.1 can be applied.

Proof: The idea behind the choice of the unitaries U_n that approximate the symmetry transformation is that, in the case of continuous symmetries such as a rotation by an angle ϕ, it is possible to interpolate "smoothly" between rotations by a fixed angle in any given finite volume, and zero rotation at infinity, in such a way that there is a uniform bound on the energy involved in such a perturbation.

Claim: it suffices to take ϕ_m defined as follows:

$$\phi_m(x) = \begin{cases} \phi & \text{if } x \in \Lambda_m \\ (2 - \frac{\min\{|x_1|,|x_2|\}}{m})\phi & \text{if } x \in \Lambda_{2m} \setminus \Lambda_m \\ 0 & \text{if } x \in x \notin \Lambda_{2m} \end{cases}$$

The quantity we need to bound is the following:

$$\|U_m\delta(U_m^*) + U_m^*\delta(U_m)\| \leq \sum_{x,y} |J(x,y)| \|\Delta_{x,y}\|$$

where

$$\begin{aligned} \Delta_{x,y} &= U_x(\phi_m(x))U_y(\phi_m(y))\Phi_{x,y}U_x(\phi_m(x))^*U_y(\phi_m(y))^* \\ &\quad + U_x(\phi_m(x))^*U_y(\phi_m(y))^*\Phi_{x,y}U_x(\phi_m(x))U_y(\phi_m(y)) - 2\Phi_{x,y} \end{aligned}$$

By expanding the unitaries we can rewrite this as follows:

$$\begin{aligned} \Delta_{x,y} &= 0 + i[\phi_m(x)X_x + \phi_m(y)X_y, \Phi_{x,y}] - i[\phi_m(x)X_x + \phi_m(y)X_y, \Phi_{x,y}] \\ &\quad + 2\sum_{n\geq 1} \frac{(-1)^n}{(2n)!} \mathrm{ad}^{2n}_{\phi_m(x)X_x+\phi_m(y)X_y}(\Phi_{x,y}) \end{aligned}$$

The trick is to realize that $\Delta_{x,y}$ only depends on the differences $\phi_m(x) - \phi_m(y)$. This can be seen as follows:

$$\phi_m(x)X_x + \phi_m(y)X_y = \frac{1}{2}(\phi_m(x) + \phi_m(y))(X_x + X_y) + \frac{1}{2}(\phi_m(x) - \phi_m(y))(X_x - X_y)$$

Let us call the first term $A_{x,y}$ and the second term $B_{x,y}$. Then, it is easily checked that $A_{x,y}$ and $B_{x,y}$ commute. Hence,

$$\mathrm{ad}_{\phi_m(x)X_x+\phi_m(y)X_y} = \mathrm{ad}_{A_{x,y}} + \mathrm{ad}_{B_{x,y}}$$

with $\mathrm{ad}_{A_{x,y}}$ and $\mathrm{ad}_{B_{x,y}}$ commuting as well. By assumption we have $\mathrm{ad}_{A_{x,y}}(\Phi_{x,y}) = 0$. Using these properties we can derive the following estimate for $\|\Delta_{x,y}\|$:

$$\|\Delta_{x,y}\| \leq 2\sum_{n\geq 1} \frac{1}{(2n)!} \left(\frac{\phi_m(x) - \phi_m(y)}{2}\right)^{2n} \|\mathrm{ad}^{2n}_{B_{x,y}}(\Phi_{x,y})\| \tag{2.2}$$

Since $d = 2$, we have $|\Lambda_m| = (2m+1)^2$. Also not that $|\phi_m(x) - \phi_m(y)| \leq |\phi|/m$. Therefore, the sum over x can be estimated by

$$\begin{aligned} \|\Delta_{x,y}\| &\leq 2 \sum_{|x|\leq 2m, y\in\mathbb{Z}^2} |J(x,y)| \left(\frac{|x-y|}{2m}\right)^2 \sum_{n\geq 1} \frac{1}{(2n)!}(2\|B_{x,y}\|)^{2n}\|\Phi_{x,y}\| \\ &\leq 4 \sum_{|x|\leq 2m, y\in\mathbb{Z}^2} |x-y|^2 |J(x,y)| \frac{\|B_{x,y}\|^2}{(2m)^2} e^{4|\phi|\|B_{x,y}\|^2} \\ &\leq \text{constant} \times \sup_x \sum_{y\in\mathbb{Z}^2} |x-y|^2 |J(x,y)| \end{aligned}$$

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One can obtain a similar condition on $J(x,y)$ that excludes continuous symmetry breaking in one dimension. For this, and some other applications, see Homework # 4.