Quantum Spin Systems and Finitely Correlated States

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Abstract: This is the first part of two. We give a general introduction to the theory and applications of Quantum Markov States, also known as Finitely Correlated States. The main topics are: 1) Quantum spin systems; 2) Finitely Correlated States and Valence Bond Solid models; 3) The spectral gap; 4) Symmetry and Quantum Symmetry; 5)The XXZ Heisenberg Model; 6) The Density Matrix Renormalization Group.

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1 Introduction

These are lecture notes, not a comprehensive review article. The aim is to give an overview of the theory of finitely correlated states and their applications, as well as some spin-off of ideas that have their origin in the study of finitely Correlated states but that have proved useful in a broader context. These notes are not a substitute for the original papers, but they can be used as a guide to the literature.

The topics covered in this first part are: Quantum Spin Systems, Finitely Correlated States, Valence Bond Solid models, and Valence Bond States.

Quantum Markov Chains have many origins. Not surprisingly, there are various situations in the physical world that can be effectively described quantum Markov chains (and more generally, quantum Markov processes or fields). It has been noted over the years that the common mathematical framework describing markovianity in classical probability can be "quantized" in more than one way. Or, as some would say, there is not a unique way to introduce markovianty in a non-commutative setting. One definition, the Quantum Markov Chains, also known as Finitely Correlated States, have proved to be very useful in the study of quantum spin models. This is the main subject of these lectures. Without going into detail, let us summarize the double origin of Finitely Correlated States (FCS). For more information on other developments we refer to the lecture notes by Accardi [Acc00] and Cecchini [Cec92].

In his PhD thesis [Acc73] (see also [Acc81, AF83]), Accardi introduced a definition of markovianity for quantum chains and provided a construction that produces non-product states with this property. Earlier attempts to introduce a conditional expectation map for quantum systems and to use it to define markovianity had failed in the sense that they did not produce interesting examples. E.g., the existence of the so-called Umegaki conditional expectation [Ume54] implies a product structure. Therefore, the fact that the construction of Accardi produced non-product examples was a remarkably fact which, however, did not attract much attention until later.

In an entirely different context and independently, Affleck, Kennedy, Lieb, and Tasaki [ALKT87, AKLT88] introduced some quantum spin Hamiltonians for which they could construct the exact ground states. The most important of their models is a spin 1 chain, now called the AKLT model, which was the first rigorous example of a quantum spin chain with a ground state that has all the characterizing properties of the Haldane phase [Hal83, AL86]. They proved that the ground state of the AKLT model is unique, has exponentially decaying (yet non-trivial) correlations and that there is a non-vanishing gap in the spectrum above the ground state. This work generated considerable interest in the physics literature.

Fannes, Nachtergaele, and Werner [FNW89b] noticed that the ground state of the AKLT model was an example of a simple generalization of Accardi's construction of quantum Markov chains, which they called Finitely Correlated States. With this generalization, it became possible to develop a coherent theory of quantum Markov chains [FNW92c, FNW92a, FNW94, Mat98], and led to several new applications in the study of quantum spin models [FNW92b, FNW92d, Nac92, FNW96, Nac96, Mat01] and beyond [HP94, BJ97, BJKW00].

2 Quantum Spin Systems

2.1 Basic Definitions

We will consider quantum spin systems on the regular cubic lattice $\mathbb{Z}^\nu$. Mostly, we will study the one-dimensional case ($\nu = 1$), in which case one speaks of *quantum spin chains.* The elements of the lattice, $x \in \mathbb{Z}^\nu$, are called *sites.* At each site x we consider a quantum system with finite-dimensional state space $\mathcal{H}_x \cong \mathbb{C}^d$, $d \geq 2$. Typical examples of importance in physics are the quantum spin systems. In that case the d-dimensional irreducible representation of $SU(2)$, also called the spin S representation with $S = (d-1)/2$, will play a fundamental role. Next, we will recall some basic definitions, mainly to fix the notation. For more details one should consult [BR97].

For every finite subset $\Lambda \subset \mathbb{Z}^\nu$, the Hilbert space of states of the system is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$, and the algebra of observables is $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. We often use the abbreviation $\mathcal{A}_x = \mathcal{A}_{\{x\}}$. Clearly, $\mathcal{A}_x \cong M_d$, the $d \times d$ matrices with complex entries, and $\mathcal{A}_\Lambda = (\otimes \mathbb{C}^d)^{|\Lambda|}$, where $|\Lambda|$ denotes the number of sites in Λ, and if $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $\mathcal{A}_{\Lambda\cup\Lambda} \cong \mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda_2}$. If $\Lambda_0 \subset \Lambda$, there is a natural embedding $j_{\Lambda_0,\Lambda} : \mathcal{A}_{\Lambda_0} \to \mathcal{A}_\Lambda$, defined by tensoring with the identity element of $\mathcal{A}_{\Lambda\setminus\Lambda_0}$. We will often use the abbreviation $j_x = j_{\{x\},\Lambda}$.

The embeddings $j_{\Lambda_0,\Lambda}$ are isometric or the usual operator norm. Hence, the direct limit $\mathcal{A}_{\rm loc} = \bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathcal{A}_\Lambda$, is naturally equipped with a norm, and we can consider the C*-algebra $\mathcal{A}_{\mathbb{Z}^\nu}$ obtained by completing $\mathcal{A}_{\rm loc}$ in the topology of this norm. This is the algebra of *quasi-local observables*, while $\mathcal{A}_{\rm loc}$ is called the algebra of *local observables.*

In applications, the lattice $\mathbb{Z}^\nu$ will be thought of as space. In the case $\nu = 1$, one could equally well choose to think of it as discrete time. For each $x \in \mathbb{Z}^\nu$, τ_x is the unique automorphism satisfying

$$\tau_x(j_y(A)) = j_{x+y}(A), y \in \mathbb{Z}^\nu, A \in M_d \quad .$$

The τ_x form a representation of the additive group $\mathbb{Z}^\nu$, and are called space translations.

The innate dynamics of a quantum spin system is determined by a set of *local Hamiltonians.* In most cases these local Hamiltonians are given in terms of a *finite-range interaction* $h = h^* \in \mathcal{A}_{\Lambda_0}$, for some finite $\Lambda \subset \mathbb{Z}^\nu$. E.g., we will often consider nearest-neighbor interactions determined by $h \in M_d \otimes M_d$. For arbitrary finite $\Lambda \subset \mathbb{Z}^\nu$, one defines

$$H_\Lambda = \sum_{x, j_x(\Lambda_0)\subset\Lambda} j_x(h)$$

The local Hamiltonians give rise to local dynamics:

$$\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad ,$$

for all $A \in \mathcal{A}$, $t \in \mathbb{R}$. It can be shown that, for all $A \in \mathcal{A}_{\rm loc}$, $\lim_{\Lambda\uparrow\mathbb{Z}^\nu} \alpha_t^\Lambda(A) =: \alpha_t(A)$, exists, uniformly on compacts in t, and that the α_t thus defined form a strongly continuous one-parameter group of C^*-automorphisms of $\mathcal{A}$. The limit $\lim_{\Lambda\uparrow\mathbb{Z}^\nu}$, can simply be taken over any increasing sequence of Λ, such that Λ eventually contains any finite set.

The generator of the dynamics $\{\alpha_t\}_{t\in\mathbb{R}}$ is the unique closed extension of the derivation

$$\delta(A) := \lim_{\Lambda\uparrow\mathbb{Z}^\nu}[H_\Lambda, A], \qquad \text{for all } A \in \mathcal{A}_{\mathrm{loc}} \quad ,$$

i.e., $\mathcal{A}_{\mathrm{loc}}$ is a core for δ. This generator can be exponentiated and, of course,

$$\alpha_t(A) = e^{it\delta}(A), \quad A \in \mathcal{A} \, .$$

As an example, we consider the ferromagnetic XXZ Heisenberg model. For $x \in \mathbb{Z}^\nu$, let S^i_x, $i = 1, 2, 3$, be the standard spin-S matrices, generating a $n = (2S+1)$-dimensional irreducible unitary representation of SU(2), with $S = 1/2, 1, 3/2, \ldots$. The local Hamiltonians of the ferromagnetic XXZ Heisenberg model are given by

$$H_\Lambda = -\sum_{\substack{x,y\in\Lambda\\|x-y|=1}} \frac{1}{\Delta}(S^1_x S^1_y + S^2_x S^2_y) + S^3_x S^3_y \tag{2.1}$$

with $\Delta \geq 1$. $\Delta = +\infty$ is the Ising model. $\Delta = 1$ is the isotropic model, also called the XXX model.

This construction of the dynamics can be generalized in several ways. E.g., one can consider Hamiltonians defined in terms of a family of finite-range interactions $h_X \in \mathcal{A}_X$, $X \subset \mathbb{Z}^\nu$, which satisfy a suitable summability condition. The standard proof of the existence of the dynamics α_t can be found in [BR97] and [Sim93]. The construction of quantum spin dynamics has been extended to include certain long range and multi-body interactions. The best result to date is by Matsui [Mat93].

2.2 The ground state problem and low-lying excitations

A *state* of the quantum spin system is a linear functional ω on $\mathcal{A}$ with the properties:

$$\omega(A^*A) \geq 0, \qquad \text{for all } A \in \mathcal{A}, \quad \omega(\mathbb{1}) = 1$$

The *ground states* of a model are the solutions of the following set of inequalities:

$$\omega(A^*\delta(A)) = \lim_{\Lambda\uparrow\infty} \omega(A^*[H_\Lambda, A]) \geq 0, \qquad \text{for all } A \in \mathcal{A}_{\mathrm{loc}}$$

These inequalities express that all local excitations –created by a local observable A–, raise the energy. This is also called *Local Stability* and we will refer to this set of inequalities as LS.

For translation invariant states, i.e., states satisfying

$$\omega \circ \tau_x = \omega, \qquad \text{for all } x \in \mathbb{Z}^\nu$$

the ground states are exactly the states that minimize the energy per site:

$$\omega(h) = \inf\{\eta(h) \mid \eta \text{ translation invariant state on } \mathcal{A}\}$$

Solutions of LS are often constructed as limits as $\Lambda \to \mathbb{Z}^\nu$, of ground states of the finite-volume Hamiltonians plus boundary terms. In general, LS may have non-translation invariant solutions, e.g., describing domain walls or interfaces. E.g, the one-dimensional XXZ Heisenberg ferromagnet has non-translation-invariant ground states called kink (and antikink) states, or soliton states. There are interface ground states of the ν-dimensional XXZ model as well. See part II of these lecture notes for an overview of recent results for this model.

At low temperatures the physical behavior of the systems modelled by quantum spin Hamiltonians is determined by the ground states and the low-lying excitations above it, i.e., states with energy slightly above the ground state energy. The mathematical setting for studying the low-lying excitations is the Gel'fand-Naimark-Segal (GNS) representation, which we now describe.

Let ω be a state satisfying LS, with the generator

$$\delta(A) = \lim_{\Lambda \uparrow \mathbb{Z}^\nu} [H_\Lambda, A]$$

Then, there is a unique Hilbert space $\mathcal{H}$, a representation π of $\mathcal{A}$ on $\mathcal{H}$, and a vector $\Omega \in \mathcal{H}$, such that,

$$\begin{aligned} &\omega(A) = \langle \Omega, \pi(A)\Omega \rangle, \qquad \text{for all } A \in \mathcal{A} \\ &\pi(\mathcal{A})\Omega \text{ is dense in } \mathcal{H} \end{aligned}$$

This is called the GNS representaton of ω. In the GNS representation the dynamics $\{\alpha_t = e^{it\delta}\}_{t\in\mathbb{R}}$ is generated by a densely defined self-adjoint operator H, the *Hamiltonian* of the infinite system,

$$\begin{aligned} \pi(\alpha_t(A)) &= e^{itH}\pi(A)e^{-itH} \\ &= \lim_{\Lambda \to \mathbb{Z}^\nu} \pi(e^{itH_\Lambda} A e^{-itH_\Lambda}) \end{aligned}$$

This H depends on the ground state ω, and, in general, so does its spectrum. H can be chosen such that $H \geq 0$, and $H\Omega = 0$. It is the spectrum of this Hamiltonian H that describes the low-lying excitations above the ground state ω. Unlike the spectrum of finite-volume Hamiltonians, it depends only on ω, and only indirectly, i.e., through ω, on the boundary conditions one can impose in finite volume.

In summary, the *ground state problem* of a quantum spin model consists of the following main questions:

1. Determine the ground state energy per spin, e_0, in the thermodynamic limit. This is under quite general circumstances independent of the boundary conditions. Let ω_Λ denote a ground state of $\mathcal{H}_\Lambda$. Then, e_0 is given by

$$e_0 = \lim_{\Lambda \uparrow \mathbb{Z}^\nu} \omega_\Lambda(H_\Lambda)$$

2. The set of ground states. In particular, determine whether or not the ground state is unique. Directly related to this issue is the notion of *spontaneous symmetry breaking*. This question requires a more precise formulation and involves careful consideration of boundary conditions.

3. Structural properties of the ground states, in particular the behavior of the correlation functions such as
$$f^{ij}(x,y) = \omega(S^i_x S^j_y)$$
Usually to determine the type of asymptotic behavior is what is most interesting: converging to a non-zero constant, oscillating, converging to zero as a power law, or exponentially fast, as $|x-y| \to \infty$?

4. The first question one would like to answer about the excitation spectrum is whether there is a gap above the ground state or not. I.e., does there exist a $\gamma > 0$ such that
$$\mathrm{spec}(H) \cap (0,\gamma) = \emptyset \quad ? \tag{2.2}$$
The *exact gap* is the supremum of the set of γ's for which (2.2) holds.

3 Finitely Correlated States and Valence Bond Solid Models

3.1 Introduction

In this section we mainly be concerned with one-dimensional systems. Generalizations to higher dimensions will be mentioned, but most questions remain to be answered in that case.

A state ω on $\mathcal{A}_{\mathbb{Z}}$ is completely determined by its values on simple tensors, i.e., observables of the form $j_{[1,n],[x+1,x+n]}(A_1 \otimes \cdots \otimes A_n)$, where $x \in \mathbb{Z}$, $n \geq 1$, $[1,n] = \{1,\ldots,n\}$, $A_1,\ldots,A_n \in \mathcal{A}$, where in this section $\mathcal{A}$ will be a finite-dimensional C*-algebra, playing the role of M_d in the previous section.

If ω is translation invariant, it suffices to consider the expectations $\omega(A_1 \otimes \cdots \otimes A_n)$. If not otherwise indicated, it will be understood that an observable of the form $A_1 \otimes \cdots \otimes A_n$ belongs to $\mathcal{A}_{[1,n]}$. We will also extend the notation $\mathcal{A}_\Lambda$ for infinite sets Λ, which will always refer to the C*-subalgebra of $\mathcal{A}_{\mathbb{Z}}$ generated by $\mathcal{A}_{\Lambda_0}$, with $\Lambda_0 \subset \Lambda$.

The simplest class of states of a quantum spin chain are the product states. A translation-invariant product state ω is entirely determined by a density matrix $\rho \in M_d$ ($\rho \geq 0$, $\mathrm{Tr}\rho = 1$), as follows;
$$\omega(A_1 \otimes \cdots \otimes A_n) = \rho(A_1) \cdots \rho(A_n),$$
where $\rho(A) = \mathrm{Tr}\rho A$. Product states express independence of the variables at each site. However, interesting phenomena are typically related to correlations. One way to detect

correlations in a translation invariant state ω is to consider the linear span of the set of functionals ϕ_B on $\mathcal{A}_{[1,\infty)}$ defined by

$$\phi_B(A_1 \otimes \cdots \otimes A_n) = \omega(B \otimes A_1 \otimes \cdots \otimes A_n)$$

for $B \in \mathcal{A}_{(-\infty,0]}$. If the linear span of $\{\phi_b \mid B \in \mathcal{A}_{(-\infty,0]}\}$ is finite-dimensional, we say that ω is *finitely correlated.*

Note that if ω is a product state, the dimension of the linear span is 1. In [FNW92c], a structure theorem (Proposition 2.3) is proved that provides a characterization of finitely correlated states in terms of a finite-dimensional matrix ordered space $\mathcal{B}$ with a distinguished positive element e, a positive linear functional ρ, on $\mathcal{B}$, and a completely positive map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$, satisfying the relations

$$\mathbb{E}(\mathbb{1} \otimes e) = e \quad \text{and } \rho(\mathbb{E}(\otimes B) = \rho(B), \text{ for all } B \in \mathcal{B} \quad .$$

The finite correlated state is then given by

$$\omega(A_1 \otimes \cdots \otimes A_n) = \rho(\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(e)) \quad , \tag{3.1}$$

where, for $A \in \mathcal{A}$, $\mathbb{E}_A : \mathcal{B} \to \mathcal{B}$ is defined by $\mathbb{E}_A(B) = \mathbb{E}(A \otimes B)$. For the details we refer to [FNW92c], where the notion of matrix ordered space is discussed further. In these notes we will restrict to the case where $\mathcal{B}$ is a finite-dimensional C*-algebra, with its natural notion of positivity. Such states we call *C*-finitely correlated.* It remains an open question whether this is a real restriction, as we have no examples of finitely correlated states that are not C*-finitely correlated. It is easy to see that if one allows infinite-dimensional $\mathcal{B}$, one can always assume it is a C*-algebra, as was already remarked in [FNW92c] and exploited in [BJKW00] and [Mat01].

In the remainder of these notes *finitely correlated* will always mean C*-finitely correlated.

3.2 Construction and Main Properties

This section is based on [FNW92c, FNW94, FNW92a]. Let $\mathcal{A}_{\mathbb{Z}}$ be the algebra of local observables built on the finite-dimensional one-site algebra $\mathcal{A}$, and let $\mathcal{B}$ be another finite-dimensional C*-algebra. One looses not much by assuming that $\mathcal{A} = M_d$. Later we will see that we can restrict to $\mathcal{B} = M_k$, but considering non-square $\mathcal{B}$ are sometimes useful. Let $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$, be completely positive map that is unit preserving (also called *unital*), i.e., $\mathbb{E}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}) = \mathbb{1}_{\mathcal{B}}$. It can be shown [AHK78] that for such $\mathbb{E}$ there exists at least one state ρ on $\mathcal{B}$ such that

$$\rho(\mathbb{E}(\mathbb{1}_{\mathcal{A}} \otimes B) = \rho(B), \text{ for all } B \in \mathcal{B} \quad .$$

Suppose ρ is such a state. It is then straightforward to verify that, with these objects and properties, (3.1) defines a translation invariant state on $\mathcal{A}_{\mathbb{Z}}$. We say that ω is the finitely correlated state generated by the triple $(\mathcal{B}, \mathbb{E}, \rho)$.

It is possible to consider the same construction (3.1) with completely positive maps that are not unit preserving, but for which there is a positive $e \in \mathcal{B}$ such that $\mathbb{E}(\mathbb{1} \otimes e) = e$. It can be shown, however, that this does not lead to new states [FNW92c]. One can also show that the state ρ can be assumed to be faithful. The argument goes as follows: let P be the support projection of ρ. Using the properties of completely positive unital maps given in the Appendix one can easily show that the maps $\mathbb{E}_A$, $A \in \mathcal{A}$, leave $P\mathcal{B}P$ invariant and that $\mathbb{E}_{\mathbb{1}}(P) = P$. Therefore, the $(\tilde{\mathcal{B}}, \tilde{\mathbb{E}}, \tilde{\rho})$, with $\tilde{\mathcal{B}} = P\mathcal{B}P$, $\tilde{\mathbb{E}}(X) = P\mathbb{E}((\mathbb{1} \otimes P)X(\mathbb{1} \otimes P))P$, $\tilde{\rho} = \rho \mid \tilde{\mathcal{B}}$, generates the same state ω as $(\mathcal{B}, \mathbb{E}, \rho)$. Triples $(\mathcal{B}, \mathbb{E}, \rho)$, with $\mathcal{B} = M_k$, unit preserving $\mathbb{E}$ and faithful ρ, will be referred to as *standard triples*.

Let $\mathcal{A} = M_d$ and $\mathcal{B} = M_k$. It can be shown that all completely positive unit preserving maps $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$, are of the form

$$\mathbb{E}(X) = \sum_\alpha V_\alpha^* X V_\alpha$$

where the linear maps $V_\alpha : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$, satisfy $\sum_\alpha V_\alpha^* V_\alpha = \mathbb{1}$. The number of terms in the sum needed is at most dk [Cho75].

A particulary important class of maps $\mathbb{E}$ are the so-called *pure maps*. Let $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ be linear and isometric, i.e., $V^*V = \mathbb{1}$. Then, $\mathbb{E}(A \otimes B) = V^* A \otimes BV$, defines a *pure* completely positive unital map in the sense that there is no non-trivial decomposition $\mathbb{E} = \mathbb{E}_1 + \mathbb{E}_2$, with E_1 and $\mathbb{E}_2$ completely positive maps. Such finitely correlate states are called *purely generated*.

Before we begin our review of general properties of finitely correlate states, we will give a few basic examples. These and some other examples are discussed in greater detail in [FNW92c, FNW92b]. More examples and a discussion of their origin will be given in Section 3.4.

1. Translation invariant product states are finitely correlated. Suppose ω is such a state on $\mathcal{A}_\mathbb{Z}$, determined by a state η on $\mathcal{A} = M_d$. Then, let $\mathcal{B} = \mathbb{C}$, and define, for all $A \in M_d, \lambda \in \mathbb{C}$,
$$\mathbb{E}(A \otimes \lambda) = \lambda \eta(A), \quad \rho(\lambda) = \lambda \quad .$$
ω is purely generated iff η is a pure state.

2. The AKLT state [AKLT88, FNW92c]. The AKLT model a spin-1 chain with local Hamiltonians given by
$$H_{[a,b]} = \sum_{x=a}^{b-1} \left[\frac{1}{3} + \frac{1}{2} \vec{S}_x \cdot \vec{S}_{x+1} + \frac{1}{6} (\vec{S}_x \cdot \vec{S}_{x+1})^2 \right] \quad .$$
Its unique ground states is the finitely correlated state generated by the triple $(\mathcal{B}, \mathbb{E}, \rho)$, defined as follows. $\mathcal{B} = M_2$. $\mathbb{E}(X) = V^* X V$, with $V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$, given by
$$\begin{aligned} V|\tfrac{1}{2}\rangle &= \sqrt{\frac{1}{3}} |1\rangle \otimes |-\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}} |0\rangle \otimes |\tfrac{1}{2}\rangle \\ V|-\tfrac{1}{2}\rangle &= \sqrt{\frac{1}{3}} |-1\rangle \otimes |\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}} |0\rangle \otimes |-\tfrac{1}{2}\rangle \end{aligned}$$

where $|m\rangle$ denotes an eigenbasis of S^3. V is also the, up to a phase factor, unique isometry satisfying the following intertwining relation

$$VD^{(1/2)} = D^{(1)} \otimes D^{(1/2)}V$$

where $D^{(S)}$ is the spin-S representation of $SU(2)$. Finally, the state ρ is given by

$$\rho(B) = \frac{1}{2}\mathrm{Tr}B \quad .$$

3. The Majumdar-Ghosh state. See [MG69, FNW92c, Nac91].

4. A non-purely generated state. See [FNW92b].

5. Classical Markov chains are finitely correlated states. See [FNS92, Acc00].

3.3 Ergodic Properties

The ergodic properties of finitely correlated states can be described in great detail. Moreover, different notions of ergodicity can be proved to be equivalent for this class of states.

As before, we consider the algebra of quasi-local observables $\mathcal{A}_{\mathbb{Z}}$. The set of all translation invariant states on $\mathcal{A}_{\mathbb{Z}}$ is a *Choquet simplex*. The extremal points of this simplex are called *ergodic* states. The finitely correlated states form a *face* of this simplex. This means that the extremal elements of the set of finitely correlated states are automatically ergodic. A translation invariant state is called *clustering*, if for all $A, B \in \mathcal{A}_{\mathbb{Z}}$,

$$\lim_{n\to\infty} \omega(A\tau_n(B)) = \omega(A)\omega(B) \quad .$$

This property is also called *strongly mixing*, as opposed to *weakly mixing* which means that for all $A, B \in \mathcal{A}_{\mathbb{Z}}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} |\omega(A\tau_n(B)) - \omega(A)\omega(B)| = 0 \quad .$$

One says that ω is *completely ergodic* if it is extremal in in the larger simplices of states that are τ^k-invariant, for all $k = 1, 2, \ldots$, i.e., if ω is extremal in the set of k-periodic states for all k.

We can understand clustering by computing $\omega(A\tau_n(B))$ explicitly. Clearly, it is sufficient to consider $A, B \in \mathcal{A}_{\mathrm{loc}}$. For $A \in \mathcal{A}_{[a,b]}$ we can define $\mathbb{E}_A : \mathcal{B} \to \mathcal{B}$ by linear extension of

$$\mathbb{E}_{A_a\otimes\cdots\otimes A_b} = \mathbb{E}_{A_a} \circ \cdots \mathbb{E}_{A_b}, A_i \in \mathcal{A} \quad .$$

Then, for $a \geq 0, b \geq 1$, for $A \in \mathcal{A}_{[-a,0]}, B \in \mathcal{A}_{[1,b]}$, we see that

$$\omega(A\tau_n(B)) = \rho(\mathbb{E}_A \circ (\mathbb{E}_{\mathbb{1}})^n \circ \mathbb{E}_B(\mathbb{1})) \tag{3.2}$$

Consider the Jordan decomposition of $\mathbb{E}_{\mathbb{1}}$: $\mathbb{E}_{\mathbb{1}} = \sum_\lambda (\lambda P_\lambda + N_\lambda)$ with

$$\begin{aligned} P_\lambda P_\mu &= \delta_{\lambda,\mu} P_\lambda \\ P_\lambda N_\mu &= N_\mu P_\lambda = \delta_{\lambda,\mu} N_\lambda \end{aligned}$$

and N_λ nilpotent. As shown in the Appendix, the eigenvalues of modulus one do not have a nilpotent part. Then, we have

$$\mathbb{E}_{\mathbb{1}}^n = \sum_{\lambda, |\lambda|=1} \lambda^n P_\lambda + R_n$$

where $\|R_n\| \le c\mu^n$ for some $0 \le \mu < 1$. By using this information in (3.2) we immediately see that only the eigenvalues of modulus one, also called the *peripheral spectrum*, matter for the limiting behavior of correlations for large n. The convergence to this limiting behavior is exponentially fast with a rate determined by the next-to-largest modulus of the eigenvalues. Therefore, we now focus on the peripheral spectrum of $\mathbb{E}_{\mathbb{1}}$.

Proposition 3.1 *Let $\mathbb{P} : \mathcal{B} \to \mathcal{B}$ be a completely positive unital map, and suppose 1 is a simple eigenvalue of $\mathbb{P}$ and let ρ be the unique invariant state, and assume ρ is faithful. Then the peripheral spectrum $\pi = \{\lambda \mid \lambda \text{ is eigenvalue of } \mathbb{P}, \text{ and } |\lambda| = 1\}$, is a cyclic group, all peripheral eigenvalues are simple and the corresponding eigenvectors can be chosen to be unitary.*

Proof:
(a) Suppose $\mathbb{P}(B) = \lambda B$, for some $0 \neq B \in \mathcal{B}$. Then $\mathbb{P}(B^*) = \mathbb{P}(B)^* = \overline{\lambda} B^*$. So, for $\lambda \in \pi$, $\lambda^{-1} = \overline{\lambda} \in \pi$.
(b) Suppose $\lambda \in \pi$, and B is an eigenvector for λ. Then, by the Kadison inequality, $B^*B = \mathbb{P}(B^*)\mathbb{P}(B) \le P(B^*B)$. Therefore, $\mathbb{P}(B^*B) - B^*B \ge 0$. As ρ is invariant under $\mathbb{P}$, we have $\rho(\mathbb{P}(B^*B) - B^*B) = 0$. But ρ is faithful, therefore $\mathbb{P}(B^*B) = B^*B$. By assumption the eigenvalue 1 is simple. therefore $B^*B = c\mathbb{1}$, for some $c > 0$. We see that $U = c^{-1/2}B$ is a unitary eigenvector for λ.
(c) Now, let U be as above. Then, $\mathbb{1} = U^*U = \mathbb{P}(U^*)\mathbb{P}(U) = \mathbb{P}(U^*U) = \mathbb{1}$. By Choi's Lemma we have that $\mathbb{P}(UX) = \mathbb{P}(U)\mathbb{P}(X)$, for all $X \in \mathcal{B}$. Let V be a unitary eigenvector with eigenvalue $\mu \in \pi$, and take $X = V$. We get $\mathbb{P}(UV) = \mathbb{P}(U)\mathbb{P}(V) = \lambda\mu UV$. Therefore, $\lambda\mu \in \pi$. In particular, $\lambda^n \in \pi$, for all n. As $\mathcal{B}$ is finite-dimensional, there must exist an integer p such that $\lambda^p = 1$.
(d) If U and V are two unitary eigenvectors with the same eigenvalue $\lambda \in \pi$, then U^*V is an eigenvector with eigenvalue 1 by the properties proved above. As 1 is simple, $U^*V = c\mathbb{1}$, and U and V are proportional. Hence, λ is simple.

Together, (a), (b), (c) and (d) prove the statement. ∎

Concretely, this proposition says that if 1 is a simple eigenvalue and ρ faithful, then the peripheral spectrum is of the form

$$\pi = \{e^{n\frac{2\pi i}{p}} \mid n = 0, \ldots, p-1\}$$

A sufficient condition for ergodicity is the what we have called weak mixing, with convergence uniform in $\|A\|\,\|B\|$. We now prove that a finitely correlated state ω with a standard triple $(\mathcal{B}, \mathbb{E}, \rho)$, such that the eigenvalue 1 of $\mathbb{E}_{\mathbb{1}}$ is simple satisfies this property and, hence, is ergodic.

The main point is to compute $\mathbb{E}_{\mathbb{1}}^n$:

$$\mathbb{E}_{\mathbb{1}}^n = \sum_j \lambda_j^n P_j + n\lambda_j^{n-1} N_j + \cdots \binom{n}{k_j} \lambda_j^{n-k_j} N_j^{k_j} \quad .$$

The terms with $|\lambda_j| < 1$ vanish exponentially fast as $n \to \infty$. We already know that the peripheral spectrum does have diagonal Jordan Blocks, i.e. no nilpotent part. Except for the eigenvalue 1, the peripheral λ satisfy

$$\lambda + \lambda^2 + \cdots + \lambda^p = 0 \quad .$$

Therefore, $\|P_1 - \frac{1}{N}\sum_{n=1}^N \mathbb{E}_{\mathbb{1}}^n\| \le CN^{-1}$, for some $C > 0$, and where P_1 is the eigenprojection for the eigenvalue 1. P_1 is a rank-1 projection due to the fact that the eigenvalue 1 is simple. It is given by $P_1(B) = \rho(B)\mathbb{1}$. In combination with (3.2), this proves that ω is weakly mixing, uniformly in $\|A\|\,\|B\|$.

The converse result also holds: if a finitely correlated state is ergodic, then it is generated by a standard triple with simple eigenvalue 1 for $\mathbb{E}_{\mathbb{1}}$ [FNW92c].

This result allows us to produce examples of ergodic finitely correlated states in abundance. Non-ergodic states are also plentiful because the set of all finitely correlated states on a given $\mathcal{A}_{\mathbb{Z}}$, is convex. We leave the easy proof of the following proposition as an exercise.

Proposition 3.2 *Let ω_1 and ω_2 be a finitely correlated state on $\mathcal{A}_{\mathbb{Z}}$, generated by triples $(\mathcal{B}_1, \mathbb{E}^{(1)}, \rho_1)$ and*
$(\mathcal{B}_2, \mathbb{E}^{(2)}, \rho_2)$, respectively. Then, for $t \in [0, 1]$, $\omega = t\omega_1 + (1-t)\omega_2$, is also finitely correlated, and generated by the triple $(\mathcal{B}_1 \oplus \mathcal{B}_2, \mathbb{E}^{(1)} \oplus \mathbb{E}^{(2)}, t\rho_1 \oplus (1-t)\rho_2)$.

It is also easy to see that the eigenvalue 1 of $\mathbb{E}_{\mathbb{1}}^{(1)} \oplus \mathbb{E}_{\mathbb{1}}^{(2)}$ is degenerate: $\mathbb{1} \oplus 0$ and $0 \oplus \mathbb{1}$ are both eigenvectors.

This suggests how to identify the completely ergodic finitely correlated states, and how to decompose an arbitrary ergodic state into its periodic components.

Proposition 3.3 *Let ω be an ergodic finitely correlated state on $\mathcal{A}_{\mathbb{Z}}$, with standard triple $(\mathcal{B}, \mathbb{E}, \rho)$, and such that the peripheral spectrum of $\mathbb{E}_{\mathbb{1}}$ is the set of pth roots of unity. Then, there is a $p-$periodic state η on $\mathcal{A}_{\mathbb{Z}}$, which considered as a state on $((\mathcal{A}\otimes)^p)_{\mathbb{Z}}$, is a completely ergodic translation invariant finitely correlated state, and such that*

$$\omega = \frac{1}{p}\sum_{n=0}^{p-1} \eta \circ \tau^n$$

Proof: We will show how to obtain the state η. The rest is straightforward.

Let $\pi = \{\lambda_q = e^{q\frac{2\pi i}{p}} \mid q = 0, \dots, p-1\}$. For $\lambda \in \pi$, let U be a unitary eigenvector. Then, as shown in the proof of Proposition 3.1, U^p is an eigenvector with eigenvalue $\lambda^p = 1$. As 1 is a simple eigenvalue, U^p is a multiple of the identity. This means that, up to a phase factor which is irrelevant, the spectral decomposition of U is of the form $U = \sum_{r=1}^p \lambda_1^r P_r$, $P_r P_s = \delta_{r,s}$, $\sum_r P_r = \mathbb{1}$. By Choi's Lemma (see Appendix), we have $\mathbb{E}_{\mathbb{1}}(XU) = \mathbb{E}_{\mathbb{1}}(X)\mathbb{E}_{\mathbb{1}}(U) = \lambda \mathbb{E}_{\mathbb{1}}(X)U$. When we substitute its spectral resolution for U, we obtain

$$\sum_r \lambda_1^r \mathbb{E}_{\mathbb{1}}(XP_r) = \lambda_1 \sum_r \lambda_1^r \mathbb{E}_{\mathbb{1}}(X)P_r = \sum_r \lambda_1^r \mathbb{E}_{\mathbb{1}}(X)P_{r-1} \quad .$$

By repeating this argument for $U^q, q = 2, \dots, p$, we obtain similar relations with λ_1 replaced by λ_q. It follows that

$$\mathbb{E}_{\mathbb{1}}(XP_r) = \mathbb{E}_{\mathbb{1}}(X)P_{r-1}, r = 1, \dots, p \bmod p$$

One also has $\mathbb{E}_{\mathbb{1}}(P_r X) = P_{r-1}\mathbb{E}_{\mathbb{1}}(X)$.

Now, define $\mathcal{B}_r = P_r \mathcal{B} P_r$. Then, for $0 \le B_r \in \mathcal{B}_r$, $0 \le A \in \mathcal{A}$, we have

$$\begin{aligned}
\mathbb{E}(A \otimes B_r) &\le \|A\| \mathbb{E}_{\mathbb{1}}(P_r B_r P_r) \\
&= \|A\| P_{r-1} \mathbb{E}_{\mathbb{1}}(B_r) P_{r-1} \\
&\le \|A\| \|B_r\| P_{r-1}
\end{aligned}$$

It follows that $\mathbb{E}(A \otimes B_r) \in \mathcal{B}_{r-1}$. Extending this by linearity to all of $\mathcal{B}_r$, we obtain $\mathbb{E}_A(\mathcal{B}_r) \subset \mathcal{B}_{r-1}$.

Define,

$$\tilde{\mathcal{B}} = \bigoplus_{r=1}^p \mathcal{B}_r \quad .$$

We have proved that $\tilde{\mathcal{B}}$ is invariant under $\mathbb{E}_A$. The pth iteration of $\mathbb{E}$ defines a completely positive unital map

$$\mathbb{E}^p : \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{p} \otimes \mathcal{B} \to \mathcal{B} \quad ,$$

with the property that $\mathbb{E}_A^p(\mathcal{B}_r) \subset \mathcal{B}_r$. This map defines η as a translation invariant finitely correlated state on the algebra $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$.

It is a matter of direct computation to check that η has the properties stated in the proposition.

∎

For more results, in particularly necessary and sufficient conditions for ergodicity in terms of properties of the generating triples, we refer to [FNW92c, FNW94, HP94]. The non-trivial part of this subject is caused by the fact that the generating triple of a finitely correlated state is highly non-unique.

Next, we consider pure finitely correlated states. A state ω is called *pure* if it has no non-trivial decompositions as a convex combination, i.e., if $\omega = t\eta_1 + (1-t)\eta_2$, with $t \in (0,1)$, and states η_1, η_2, then $\eta_1 = \eta_2 = \omega$.

It will turn out that pure finitely correlated states can be given in terms of triples with a pure completely positive unital map. For such triples there is a good notion of minimality.

Definition 3.4
(i) A standard triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating a finitely correlated state ω is called minimal*, if there is no proper subalgebra of $\mathcal{B}$ containing $\mathbb{1}$, which is invariant under all the maps $\mathbb{E}_A$, $A \in \mathcal{A}$.*
(ii) Two standard triples $(\mathcal{B}_1, \mathbb{E}^{(1)}, \rho_1)$ and $(\mathcal{B}_2, \mathbb{E}^{(2)}, \rho_2)$, are called isomorphic *if there is a C^*-isomorphism $\tau : \mathcal{B}_1 \to \mathcal{B}_2$, such that*

$$\mathbb{E}^{(2)} \circ (\mathrm{id}_{\mathcal{A}} \otimes \tau) = \tau \circ \mathbb{E}^{(1)} \quad ,$$

and $\rho_2 \circ \tau = \rho_1$.

This is a useful definition in the case of purely generated ω, as the following theorem shows.

Theorem 3.5 *Let ω be a purely generated finitely correlated state. Then, up to isomorphism, there is only one minimal triple that generates ω.*

There is no such theorem in the general case where $\mathbb{E}$ is not assumed to be pure. This seems due to the fact that the restriction to $\mathcal{B}$ that are C*-algebras is not entirely natural. This may ba a hint that matrix ordered spaces are needed after all. See [FNW94] for a discussion.

Note that, as before, finitely correlated in these notes always means C*-finitely correlated. Recall that the *mean entropy*, $s(\omega$, of a translation invariant state is defined as follows:

$$s(\omega) = -\lim_{N\to\infty} \frac{1}{2N+1} Tr D_{[-N,N]} \log D_{[-N,N]} \quad ,$$

where D_Λ denotes the density matrix of ω restricted to $\mathcal{A}_\Lambda$.

Theorem 3.6 *Let ω be a finitely correlated state. Then, the following are equivalent:*
(i) ω is pure.
(ii) $s(\omega) = 0$ and ω is clustering, i.e., for $A, B \in \mathcal{A}_\Lambda$:

$$\lim_{n\to\infty} \omega(A \otimes \underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{n \text{ times}} \otimes B) = \omega(A)\omega(B) \quad .$$

(iii) ω is purely generated and $\mathbb{E}_{\mathbb{1}}$ has trivial peripheral spectrum, i.e., the only eigenvector of $\mathbb{E}_{\mathbb{1}}$ with eigenvalue of modulus 1, is $\mathbb{1}$.
(iv) ω is generated by a minimal triple $(\mathcal{B}, \mathbb{E}, \rho)$, with $\mathcal{B} = M_k$, and $\mathbb{E}(X) = V^* X V$ with an isometry $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$.

This theorem is proved in [FNW94]. The entropy condition appears here as an equivalent property for purely generatedness [FNW94]. For more interesting results concerning the entropy and relative entropy for finitely correlated states, see [FNW92b, HP94]. Note that, in (iv), the condition $\mathcal{B} = M_k$, has non-trivial content. It excludes the existence of decompositions into periodic, non-translation invariant states.

The theorem above allows us the construct a large number of translation invariant pure states on $\mathcal{A}$. In fact, one can show that the set of all pure finitely correlated states is w^*-dense in the set of all translation invariant states of $\mathcal{A}_{\mathbb{Z}}$, with $\mathcal{A} = M_d$ [FNW92a]. This is in sharp contrast with the classical case (abelian $\mathcal{A}$). There, the translation invariant pure states are just the Dirac measures on translation invariant configurations.

In the next section we discuss the relation of (pure) finitely correlated states with the ground state problem for quantum spin models.

3.4 Valence Bond Solid Models

This section is based on [FNW92c, Nac92, Nac96, Mat98]. The main result we want to discuss in this section is the following: for any translation invariant finite set $\mathcal{S}$ of pure finitely correlated state on $\mathcal{A}_{\mathbb{Z}}$, there exists a finite-range interaction h such that the set of ground states of the translation invariant quantum spin model defined by h is exactly the convex hull of $\mathcal{S}$. In particular, every pure finitely correlated state is the unique ground state of a finite range quantum spin model. Instead of copying the proofs given in [FNW92c, Nac96], we will explain why this is true. In particular, we will describe a property of finitely correlated states that is the basis for this result: the *intersection property* of the local support spaces. We will also discuss the *commutation property* of the local support projections. The intersection property is responsible for the existence of a finite range interaction with the desired properties. The commutation property will play an essential role in proving the existence of a spectral gap for the resulting quantum spin models (see Part II of these lecture notes). In order to treat the case of arbitrary sets of pure finitely correlated states, one also needs the *orthogonality property* of the support spaces of distinct finitely correlated states. Let us introduce the *local support spaces* first.

Let us start with a pure finitely correlated state ω, generated by a standard triple $(\mathcal{B}, \mathbb{E}, \rho)$, $\mathbb{E} = V^* \cdot V$, with $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ an isometry, and with the property that $\mathbb{E}_{\mathbb{1}}$ has trivial peripheral spectrum. As before, let D_Λ denote the density matrix of $\omega \mid \mathcal{A}_\Lambda$. As ρ is faithful, its density matrix has k eigenvalues $\rho_1 > 0, \ldots, \rho_k > 0$. By translation invariance $D_{[a,b]}$ depends only on $b - a$. It is easy to see that $\mathrm{rank} D_{[a,b]} \leq k^2$, and one can prove that

$$m_0 = \inf\{m \geq 0 \mid \mathrm{rank} D_{[1,m]} = k^2\} < +\infty \tag{3.3}$$

and that $\mathrm{rank} D_{[1,m]} = k^2$, for all $m \geq m_0$. Moreover, as $m \to \infty$, the non-zero eigenvalues of $D_{[1,m]}$ converge to the numbers $\rho_i \rho_j$, $1 \leq i, j \leq k$. Denote by $\mathcal{G}_\Lambda \subset \mathcal{H}_\Lambda$ the support space of $\omega \mid \mathcal{A}_\Lambda$, i.e., $\mathcal{G}_\Lambda$ is the range of D_Λ. As ω is pure, the vectors in $\mathcal{G}_{[a,,b]}$, must "look the same" in the interior of $[a, b]$. This is expressed in the following property of normalized eigenvectors

$\psi_{ij}^{\Lambda}, 1 \le i, j \le k$, of D_Λ:

$$\lim_{a\to-\infty, b\to+\infty} \langle \psi_{ij}^{[a,b]}, A\psi_{kl}^{[a,b]} \rangle = \omega(A)\delta_{i,k}\delta_{j,l} \quad , \tag{3.4}$$

for all $A \in \mathcal{A}_{\mathrm{loc}}$.

Theorem 3.7 (Intersection Property, [FNW92c]) *There exists an integer, l_0, such that for all $l \ge l_0$, and all $a, b \in \mathbb{Z}$ such that $b - a \ge l$, one has*

$$\mathcal{G}_{[a,b]} = \bigcap_{k=0}^{b-a-l+1} \mathcal{H}_{[a,a+k-1]} \otimes \mathcal{G}_{[a+k,a+k+l-1]} \otimes \mathcal{H}_{[a+k+l,b]} \tag{3.5}$$

Here, l_0 can always taken to be equal to $m_0 + 1$, with m_0 defined in (3.3).

In some, but not all, cases (3.5) also holds with $l = m_0$. The following equivalent form of (3.5) is sometimes useful: for all $l \ge l_0$, all $a, b, c \in \mathbb{Z}$ such that $a \le b, b + l \le c$, one has

$$\mathcal{G}_{[a,c]} = \mathcal{G}_{[a,b+l]} \otimes \mathcal{H}_{[b+l+1,c]} \cap \mathcal{H}_{[a,b]} \otimes \mathcal{G}_{[b+1,c]}$$

It is now clear how the finite-range interaction can be defined. Let $h = h^*$ be a non-negative element of $\mathcal{A}_{[1,l]}$, with $l \ge l_0$, and such that $\ker h = \mathcal{G}_{[1,l]}$, and define $H_{[a,b]}$ by

$$H_{[a,b]} = \sum_{x,[x,x+l-1]\subset[a,b]} j_{[1,l],[a,b]}(h)$$

Then, due to the intersection property we will have

$$\ker H_{[a,b]} = \bigcap_{x,[x,x+l-1]\subset[a,b]} \ker j_{[1,l],[a,b]}(h) = \mathcal{G}_{[a,b]}.$$

From here it is straightforward to show that ω is a ground state of the model with interaction h. (3.4) can be used to show that ω is the only state satisfying

$$\omega(\tau_x(h)) = 0, \text{ for all } x \in \mathbb{Z} \quad .$$

The converse of this result was obtained by Matsui [Mat98], i.e., if local Hamiltonians with the properties described above exist, and ω is clustering, then ω is a pure finitely correlated state.

In the next section we will also need the following property of the ground state projections G_Λ on the spaces $\mathcal{G}_\Lambda$: there exists a $C > 0$ and a $0 \le \lambda < 1$ such that for all $l \ge l_0$, $a, b, c \in \mathbb{Z}, a \le b, b + l \le c$, one has

$$\|G_{[a,c]} - (G_{[a,b+l]} \otimes \mathbb{1}_{[b+l+1,c]})(\mathbb{1}_{[a,b]} \otimes G_{[b+1,c]})\| \le C\lambda^l \tag{3.6}$$

We call this the *commutation property* of the ground state projections. Indeed, (3.6) implies that the ground state projections for two intervals that have a large intersection, almost commute. This property is related with the "good factorization property" proved in [HP94] for GVBS states and similar to the factorization property for some classical partition functions, given in [OP90].

In order to generalize the construction of the Hamiltonians to the case of an arbitrary finite set of finitely correlated states, one new ingredient is needed. Let ω_1 and ω_2 be two distinct pure finitely correlated states on $\mathcal{A}_{\mathbb{Z}}$, and let ψ^{Λ}_{ij} and ϕ^{Λ}_{kl} be the normalized eigenvectors of their perspective local density matrices. Then,

$$\lim_{a\to-\infty, b\to+\infty} \langle \psi^{[a,b]}_{ij}, A\phi^{[a,b]}_{kl}\rangle = 0 \quad , \tag{3.7}$$

for all $A \in \mathcal{A}_{\mathrm{loc}}$. It seem reasonable then that one can find an interaction with kernel equal to the direct sum of the local support spaces of the finite set of states. We refer to [Nac96] for the details.

3.5 Valence Bonds Solid States and Generalizations

This section is based on [FNW92c, FNW92d].

In [AKLT88] the ground state of the AKLT model was given in terms of a construction reminiscent of describing the electron state of a molecule with valence bonds. Hence, the name Valence Bond Solid (VBS) state. It turns out that every finitely correlated state can be obtained in a similar way. Thus, we have an alternative construction for finitely correlated states. In the one-dimensional situation, which we have been studying so far, the valence bond construction may not offer any particular advantages, but it does seem to provide a way to generalize the notion of quantum markov states to higher dimensions.

The general VBS construction is given in terms of the following objects: two auxiliary finite-dimensional C*-algebras $\mathcal{B}$ and $\overline{\mathcal{B}}$, a completely positive map $\mathbb{F} : \mathcal{A} \to \mathcal{B} \otimes \overline{\mathcal{B}}$, and a state $\Phi : \overline{\mathcal{B}} \otimes \mathcal{B} \to \mathbb{C}$, which have to satisfy the compatibility conditions

$$(\mathrm{id}_{\mathcal{B}} \otimes \Phi)(\mathbb{F}(\mathbb{1}_{\mathcal{A}}) \otimes \mathbb{1}_{\mathcal{B}}) = \mathbb{1}_{\mathcal{B}} \qquad \text{and} \quad (\Phi \otimes \mathrm{id}_{\overline{\mathcal{B}}})(\mathbb{1}_{\overline{\mathcal{B}}} \otimes \mathbb{F}(\mathbb{1}_{\mathcal{A}})) = \mathbb{1}_{\overline{\mathcal{B}}} \quad . \tag{3.8}$$

On any n consecutive sites a state ω is then defined by

$$\omega(A_1 \otimes \cdots \otimes A_n) = \underbrace{\Phi \otimes \cdots \Phi}_{n+1} \, (\mathbb{1}_{\overline{\mathcal{B}}} \otimes \mathbb{F}(A_1) \otimes \cdots \mathbb{F}(A_n) \otimes \mathbb{1}_{\mathcal{B}}) \quad . \tag{3.9}$$

The compatibility conditions ensure that the states on $\mathcal{A}_{[1,n]}$ defined above determine a translation invariant state on $\mathcal{A}_{\mathbb{Z}}$. The states which can be obtained in this way are called a *valence bond* states. The construction can be visualized as follows

$$\begin{array}{ccccccccccc}
 & & \mathcal{A} & \otimes & \mathcal{A} & \otimes & \mathcal{A} & \otimes\cdots\otimes & \mathcal{A} & & \\
 & & \Big\downarrow{\scriptstyle \mathbb{F}} & & \Big\downarrow{\scriptstyle \mathbb{F}} & & \Big\downarrow{\scriptstyle \mathbb{F}} & & \Big\downarrow{\scriptstyle \mathbb{F}} & & \\
1\!\!1_{\overline{\mathcal{B}}} & \otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes\cdots\otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes & 1\!\!1_{\mathcal{B}} \\
 & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi} & & & & \Big\downarrow{\scriptstyle \Phi} & \\
 & \mathbb{C} & \otimes & \mathbb{C} & \otimes & \mathbb{C} & & \otimes\cdots\otimes & & \mathbb{C} & \equiv \mathbb{C}
\end{array}$$

The connection to the class of finitely correlated states states is made in the following Proposition.

Proposition 3.8 *Every finitely correlated state state is a valence bond state and conversely. Moreover, in the representation of a valence bond state we may take $\mathcal{B} \cong \overline{\mathcal{B}} \cong M_k$, and Φ to be a pure state with faithful restriction to either factor.*

Proof: Given a valence bond state, we define

$$\mathbb{E}(A \otimes B) = (\mathrm{id}_{\mathcal{B}} \otimes \Phi)(\mathbb{F}(A) \otimes B) \qquad \text{and} \qquad \rho(B) = \Phi(1\!\!1_{\overline{\mathcal{B}}} \otimes B) \quad .$$

Then the compatibility conditions for $\mathbb{F}$ and Φ become those for $\mathbb{E}$ and ρ, and one checks by induction on n, that ω is generated by $\mathbb{E}, \rho$.

The proof of the converse and remaining statements, though a bit tedious, is not hard. An essential step is to consider a finitely correlated state $\tilde{\omega}$, the dilation of ω, which is a purely generated finitely correlated state on an algebra $\tilde{\mathcal{A}}_{\mathbb{Z}}$, with one-site algebra $\tilde{\mathcal{A}} = \mathcal{B}(\mathbb{C}^{\tilde{d}})$, and such that

$$\omega(A_1 \otimes \cdots \otimes A_n) = \tilde{\omega}(\pi(A_1) \otimes \cdots \otimes \pi(A_n)),$$

where $\pi : \mathcal{A} \to \mathcal{B}(\mathbb{C}^{\tilde{d}})$ is a representation of $\mathcal{A}$. See [FNW92c] for the complete proof. ∎

As an example, we now consider the valence bond description of the ground state of the AKLT model, i.e., we will specify the maps $\mathbb{F}$ and Φ appearing in (3.9). Let $\mathcal{B} = \overline{\mathcal{B}} = M_2$. In order to define $\mathbb{F}$ we introduce a linear map $W : \mathbb{C}^3 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ defined by

$$\begin{aligned}
W|1\rangle &= \frac{2}{\sqrt{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle \\
W|0\rangle &= \frac{1}{\sqrt{3}}(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |-\tfrac{1}{2}, \tfrac{1}{2}\rangle) \\
W|-1\rangle &= \frac{2}{\sqrt{3}}|-\tfrac{1}{2}, -\tfrac{1}{2}\rangle
\end{aligned}$$

W equals $\frac{2}{\sqrt{3}}\times$ the embedding of $\mathbb{C}^3$ as state space of a spin 1 into the state space $\mathbb{C}^2 \otimes \mathbb{C}^2$ of two spin 1/2's. The maps $\mathbb{F}$ and Φ are given by:

$$\mathbb{F}(A) = WAW^* \qquad \Phi(B_1 \otimes B_2) = \langle\phi|B_1 \otimes B_2\phi\rangle$$

where ϕ is the singlet vector $\sqrt{1/2}(|\frac{1}{2},-\frac{1}{2}\rangle - |-\frac{1}{2},\frac{1}{2}\rangle)$. We can now verify the compatibility conditions (3.8) and thus obtain a valence bond state.

To verify that this state coincides with the AKLT ground state defined as a finitely correlated state in (2), one has to compute the following maps:

$$V = (W^* \otimes \mathbb{1}_{M_2})\,(\mathbb{1}_{M_2} \otimes \phi), \qquad \tfrac{1}{2}\mathrm{Tr}B = \Phi(\mathbb{1}_{M_2} \otimes B) \quad B \in M_2$$

The reader can check that V is the isometry which defines the completely positive map $\mathbb{E} = V^* \cdot V$, used in the construction of the AKLT state. It is then easy to obtain the FCS3.1 formula for ω from (3.9).

The main interest of the valence bond construction is that it generalizes immediately to graphs other than the one-dimensional chain: an algebra $\mathcal{A}_i$ is then associated to any vertex "i" of the graph, and an algebra $\mathcal{B}_{ij}$ to each directed edge. then needs two kinds of completely positive maps: map $\mathbb{F}_i$ taking each $\mathcal{A}_i$ into the tensor product of the outgoing edge algebras, and "contractions" $\mathcal{B}_{ij} \otimes \mathcal{B}_{ji} \to \mathbb{C}$. With each set Λ of vertices one associates the observable algebra $\mathcal{A}_\Lambda = \bigotimes_{i\in\Lambda} \mathcal{A}_i$ and the *outgoing edge algebra* $\mathcal{B}_{\partial\Lambda} = \bigotimes_{i\in\Lambda, j\notin\Lambda} \mathcal{B}_{ij}$. Clearly, every state on $\mathcal{B}_{\partial\Lambda}$ is transformed via the $\mathbb{F}_i$ and the contractions into a state on $\mathcal{A}_\Lambda$. In order to get an explicit definition of a state on the infinite system one either has to show the existence of a unique limit state, independent of the choice of states for $\mathcal{B}_{\partial\Lambda}$ (which serve as "boundary conditions"), or one has to find and verify appropriate compatibility conditions for these states. Both problems appear to be highly nontrivial. An example of a result of the first kind for a two-dimensional example have been obtained in [KLT88]. Some general answers can be given in the case of the Bethe lattice (Cayley tree), see [FNW92d].

4 The Spectral Gap

4.1 A General Estimate

This section is based on [Nac96]

4.2 Application to VBS Models

This section is based on [FNW92c, Nac96]

4.3 Other Applications

This section is based on [Nac96, CNS]

5 Symmetry and Quantum Symmetry

5.1 Finitely Correlated States with Symmetries

This section is based on [FNW92c, FNW94, FNW, Nac96, Mat01]

5.2 Quantum Group Symmetry

This section is based on [FNW96, KN97]

6 The XXZ Heisenberg Model

This section is based on [KN97, Mat96, KN98a, BCN00, BCNS00b, BCNS00a, NS01, KNS, Sta01] See also Spitzer.

7 The Density Matrix Renormalization Group

Finding the ground state of a quantum spin system is, in general, highly non-trivial. This is closely related to the *state extension problem.* In brief, the problem is the following: suppose you are given a state ω_{Λ_0} on $\mathcal{A}_{\Lambda_0}$, and consider a finite or infinite $\Lambda \subset \mathbb{Z}$ containing Λ_0. Does there exist a state ω_Λ on $\mathcal{A}_\Lambda$ such that $\omega_\Lambda \mid \mathcal{A}_{\Lambda_0} = \omega_{\Lambda_0}$, i.e., ω_Λ is an extension of ω_{Λ_0}. See [Wer90] for a discussion of this problem.

What makes finitely correlated states so interesting is that it provides a systematic way to construct states for arbitrary finite chains that are guaranteed to be extendable to the infinite chain. This allows us to use them as variational states in a scheme for approximating the ground states of a quantum spin model. The density result of [FNW92a] mentioned earlier makes it plausible that convergent approximation schemes based on finitely correlated states can be found. Indeed, such a scheme was developed by Steven White [Whi92, Whi93].

The aim of this section is to introduce a simple version of the Density Matrix Renormalization Group algorithm and explain its relation with finitely correlated states. In particular, we will discuss the use of the Density Matrix Renormalization Group as an algorithm to decide whether a quantum spin Hamiltonian admits finitely correlated ground states. It will be obvious that many modifications and extensions of the algorithm are possible. Here, we will only discuss the basic elements as an introduction to the subject. A good starting point for more information on the state of the art and the many applications of the Density Matrix Renormalization Group is [PWK99].

7.1 The Density Matrix Renormalization Group algorithm

For simplicity we will assume that the quantum spin model we are interested in is translation invariant and has only nearest neighbor interactions. This restriction is not imposed by the method; we impose it for clarity of the presentation only. It will be convenient to label the lattice sites by $\mathbb{Z} \setminus \{0\} = \{\ldots, \bar{2}, \bar{1}, 1, 2, \ldots\}$. So, our local Hamiltonians are of the form

$$H_{[\bar{n},n]} = \sum_{x=\bar{n}}^{n-1} h_{x,x+1}$$

with $h = h^* \in \mathcal{A}_{0,1} \cong M_d \otimes M_d$. For simplicity of the notations, we will also assume that $H_{[\bar{n},n]}$ has a unique ground state, but this is not essential.

Step 1 Pick a positive integer l_0 (usually l_0 should be at least 2), and find the exact ground state of $H_{[\bar{l}_0,l_0]}$. Here, "exact" usually means calculated by a reliable numerical method such as the Lanczos diagonalization algorithm. Let's call this ground state vector ψ^1.

Step 2 Compute the density matrix $\rho_{[1,l_0]}$ of the state $\psi^{(1)}$ restricted to $\mathcal{A}_{[1,l_0]}$. Explicitly:

$$\rho_{[1,l_0]} = \mathrm{Tr}_{\mathcal{H}_{[\bar{l}_0,\bar{1}]}} |\psi^{(1)}\rangle\langle\psi^{(1)}| \ .$$

Diagonalize $\rho_{[1,l_0]}$, and call the eigenvalues $\rho_1^{(1)} \geq \rho_2^{(1)} \geq \cdots \geq \rho(1)_{d^{l_0}} \geq 0$, and the corresponding eigenvectors $\phi(1)_1, \phi(1)_2 \ldots$. Pick a positive integer $m_0 \leq d^{l_0}$, and define the subspace $\mathcal{K}^{(1)} \subset \mathcal{H}_{[1,l_0]}$ by

$$\mathcal{K}^{(1)} = \mathrm{Span}\{\phi_1^{(1)}, \ldots, \phi_{(m_0)}^{(1)}\} \ .$$

If $\rho_{m_0}^{(1)} = \rho_{m_0+1}^{(1)}$, there is an ambiguity here as to which vectors to choose, but generically this does not occur. If it does, a reasonable decision procedure should be implemented in the algorithm, e.g., if $\rho_{m_0}^{(1)} > 0$, one could increase the value of m_0 until $\rho_{m_0}^{(1)} \neq \rho_{m_0+1}^{(1)}$, and if $\rho_{m_0}^{(1)} = 0$, the obvious choice would be to decrease m_0 until $\rho_{m_0}^{(1)} \neq 0$. We also define

$$\bar{\mathcal{K}}^{(1)} = \mathrm{Span}\{R\phi_1^{(1)}, \ldots, R\phi_{(m_0)}^{(1)}\} \ ,$$

where R denotes the unitary in $\mathcal{A}_{[1,l_0]}$ defined by $R(u_1 \otimes \cdots \otimes u_{l_0}) = u_{l_0} \otimes \cdots \otimes u_1$, for $u_1, \ldots, u_{l_0} \in \mathbb{C}^d$. In the sequel, we will use the same notation R for the similarly defined unitaries implementing the reflection on arbirary intervals $[a,b] \subset \mathbb{Z}$.

Step 3 Compute the projection of $H_{[\bar{l},l]}$ to $\bar{\mathcal{K}}^{(1)} \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathcal{K}^{(1)}$, with $l = l_0 + 1$, and find its ground state $\psi^{(2)}$. Let's assume this ground state is also non-degenerate. Now, go to *Step 1'* with $n = 2$.

Step 2' This step depends on $n \geq 1$. Let $l = l_0 + n - 1$. Compute the density matrix $\rho_{[1,l]}$ of the state $\psi^{(n)}$ restricted to $\mathcal{A}_{[1,l]}$. Explicitly:

$$\rho_{[1,l]} = \mathrm{Tr}_{\bar{\mathcal{K}}^{(n-1)} \otimes \mathbb{C}^d} |\psi^{(n)}\rangle\langle\psi^{(n)}| \ .$$

Diagonalize $\rho_{[1,l]}$, and call the eigenvalues $\rho_1^{(n)} \geq \rho_2^{(n)} \geq \cdots$, and the corresponding eigenvectors $\phi(n)_1, \phi(n)_2 \ldots$. Define the subspace $\mathcal{K}^{(n)} \subset \mathcal{H}_{[1,l]}$ by

$$\mathcal{K}^{(n)} = \mathrm{Span}\{\phi_1^{(n)}, \ldots, \phi_{(m_0)}^{(n)}\} \ .$$

and

$$\bar{K}^{(n)} = \mathrm{Span}\{R\phi_1^{(n)}, \ldots, R\phi_{(m_0)}^{(n)}\} \ .$$

Step 3' This step also depends on $n \geq 1$, but now $l = l_0 + n$. Compute the projection of $H_{[\bar{l},l]}$ to $\bar{\mathcal{K}}^{(n)} \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathcal{K}^{(n)}$, and find its ground state $\psi^{(n+1)}$. Now, go to *Step 2'* with n replaced by $n+1$.

Steps 2' and 3' are to be repeated in succession until some stopping criterion is satisfied. Note that Step 3' with the given value of n finds the ground state of the model projected onto a subspace of the form $\bar{\mathcal{K}}^{(n)} \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathcal{K}^{(n)}$, and the next instance of Step 2' finds the subspaces

$$\mathcal{K}^{(n+1)} \subset \mathbb{C}^d \otimes \mathcal{K}^{(n)} \subset \mathcal{H}_{[1,l_0+n-1]}, \text{ and } \bar{\mathcal{K}}^{(n+1)} \subset \bar{\mathcal{K}}^{(n)} \otimes \mathbb{C}^d \subset \mathcal{H}_{[\bar{l}_0+n-1,\bar{1}]} \quad .$$

It is easy to see that the DMRG algorithm will find the exact ground state of the model if the following two conditions are satisfied: 1) the unique infinite-volume ground state of the quantum spin model is a finitely correlated state, i.e., the model is a VBS model, and 2) we have chosen $m_0 \geq k^2$, where k is the size of the matrices in the algebra $\mathcal{B}$ appearing in the minimal triple of the state.

7.2 The Density Matrix Renormalization Group and a criterion for Valence Bond Solid models

In this section we will present an algorithm to detect Valence Bond Solid models and to compute, numerically (or, in principle, also analytically) the triple generating the finitely correlated state(s) that are the ground states of such as a model. Concretely, suppose we are considering an interaction h as above, for which we carry out the DMRG algorithm. The two questions we wish to answer are: 1) Is the true ground state of the model a finitely correlated state or not? 2) If the ground state is finitely correlated, how does one obtain a triple $(\mathcal{B}, \mathbb{E}, \rho)$, generating the state? The solution we propose is in the form of an algorithm, based on the DMRG, to compute the isometry V implementing $\mathbb{E}$. If the algorithm finishes successfully, the answer to the first question is yes, and a generating triple can be built from V.

Before we describe the algorithm itself, we first derive an explicit expression for a set of vectors spanning the local support spaces of a purely generated finitely correlated state in terms of the isometry V generating it. Let $(\mathcal{B}, \mathbb{E}, \rho)$, with $\mathbb{E} = V^* \cdots V$, $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$, be a standard triple for a finitely correlated state ω, and let ρ_L be the density matrix of $\omega \mid \mathcal{A}_{[1,L]}$. For all $A_1, \ldots, A_L \in \mathcal{A}$, ρ_l satisfies

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_{1,L]}\rho_l A_1 \otimes \cdots \otimes A_L} \\
&= \mathrm{Tr}_{\mathbb{C}^k}\, \rho V^*(\mathbb{1} \otimes V^*) \cdots (\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V^*)(A_1 \otimes \cdots A_L \otimes \mathbb{1})(\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V) \cdots (\mathbb{1} \otimes V)V \\
&= \mathrm{Tr}_{\mathbb{C}^d \otimes \cdots \mathbb{C}^d \otimes \mathbb{C}^k} V(\mathbb{1} \otimes v) \cdots (\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V)\rho V^*(\mathbb{1}^*)(\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V^*)(A_1 \otimes \cdots A_L \otimes \mathbb{1})
\end{aligned}$$

It follows that

$$\rho_L = \mathrm{Tr}_{\mathbb{C}^k} V(\mathbb{1} \otimes V) \cdots (\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V)\rho V^*(\mathbb{1}^*)(\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V^*)(A_1 \otimes \cdots A_L \otimes \mathbb{1}) \tag{7.1}$$

Note that this formula expresses ρ_L is a sum of k terms of non-negative operators each of rank $\leq k$. Hence, rank$\rho_L \leq k^2$, a fact that we have mentioned before. We have also seen that there is L_0 such that for all $L \geq L_0$, rank$\rho_L = k^2$. Assume $L \geq L_0$.

Next, we find a spanning set of vectors for ranρ_L. Let $\{e_1, \dots, e_d\}$ be a basis for $\mathbb{C}^d$, and let $v_1, \dots, v_d \in M_k$ be such that

$$Vu = \sum_{i=1}^{d} e_i \otimes v_i u, \text{ for all } u \in \mathbb{C}^k \quad .$$

Let $V_L : \mathbb{C}^k \to \underbrace{\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{L} \otimes \mathbb{C}^k$ denote the isometry defined by

$$V_L = V(\mathbb{1} \otimes V) \cdots (\underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{L-1} \otimes V)$$

Then, for any $u \in \mathbb{C}^k$, we have

$$V_L u = \sum_{i_1, \dots, i_L = 1}^{d} e_{i_1} \otimes \cdots e_{i_L} \otimes v_{i_1} \cdots v_{i_L} u$$

Starting from (7.1), in the basis $e_{i_1} \otimes \cdots e_{i_L}$, we can now express the matrix elements of ρ_L. It will be convenient to use an orthonormal basis $\{|i\rangle \mid i = 1, \dots, k\}$ of $\mathbb{C}^k$, that diagonalizes ρ. We have

$$\rho_L = \sum_{i,j=1}^{k} \langle j|\rho|j\rangle \langle i|V_L|j\rangle \langle j|V_L^*|i\rangle \quad .$$

Appendix

In this appendix, let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C*-algebra of operators on a Hilbert space $\mathcal{H}$.

Lemma A.1 (Kadison inequality, 2-positivity inequality) *If $\mathbb{P} : \mathcal{A} \to \mathcal{A}$ is a 2-positive linear map, then*

$$\mathbb{P}(A^*)\mathbb{P}(A) \leq \mathbb{P}(A^*A), \text{ for all } A \in \mathcal{A} \quad .$$

Lemma A.2 (Choi) *Suppose $\mathbb{P}$ is a completely positive map such that $\mathbb{P}(\mathbb{1}) = \mathbb{1}$. If $A \in \mathcal{A}$ is such that $\mathbb{P}(A^*)\mathbb{P}(A) = \mathbb{P}(A^*A)$, then*

$$\mathbb{P}(BA) = \mathbb{P}(B)\mathbb{P}(A), \text{ for all } B \in \mathcal{A} \quad .$$

Corollary A.3 *A CPU map $\mathbb{P}$ is a contraction, i.e.,*

$$\|\mathbb{P}(A)\| \leq \|A\| \quad .$$

Corollary A.4 *The spectral radius of a completely positive unital map $\mathbb{P}$ is 1, and the Jordan block of eigenvalues λ of $\mathbb{P}$ with $|\lambda| = 1$, have no nil-potent part.*

Proof: Consider the Jordan decomposition of $\mathbb{P}$: $\mathbb{P} = \sum_\lambda (\lambda P_\lambda + N_\lambda)$ with

$$\begin{aligned} P_\lambda P_\mu &= \delta_{\lambda,\mu} P_\lambda \\ P_\lambda N_\mu &= N_\mu P_\lambda = \delta_{\lambda,\mu} N_\lambda \end{aligned}$$

and N_λ nilpotent. Then, we have

$$\mathbb{E}_{\mathbb{1}}^n = \sum_{\lambda, |\lambda|=1} (\lambda P_\lambda + N_\lambda)^n + R_n$$

where $\|R_n\| \leq c\mu^n$ for some $0 \leq \mu < 1$. Consider one of the eigenvalues λ with $|\lambda| = 1$, and suppose k is such that $N_\lambda^k \neq 0$, $N_\lambda^{k+1} = 0$. Then,

$$\begin{aligned} (\lambda P_\lambda + N_\lambda)^n &= \lambda^n P_\lambda + n\lambda^{n-1} N_\lambda + \cdots + \binom{n}{k} \lambda^{n-k} N_\lambda^k \\ &= \lambda^{n-k} \left[\lambda^k P_\lambda + n\lambda^{k-1} N_\lambda + \cdots + \binom{n}{k} N_\lambda^k \right] \end{aligned}$$

By taking a matrix element for vectors such that the term with N_λ^k does not vanish, we obtain, apart form the constant in front of the sum, which is a function of constant modulus, a polynomial in n of degree k. This can be bounded only if $k = 0$. ■

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