Sharp Ellipsoid Embeddings and Toric Mutations

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ABSTRACT: This article introduces a new method to construct volume-filling symplectic embeddings of 4-dimensional ellipsoids by employing polytope mutations in toric and almost-toric varieties. The construction uniformly recovers the sharp sequences for the Fibonacci Staircase of McDuff-Schlenk, the Pell Staircase of Frenkel-Müller and the Cristofaro-Gardiner-Kleinman’s Staircase, and adds new infinite sequences of sharp ellipsoid embeddings. In addition, we initiate the study of symplectic tropical curves for almost-toric fibrations and emphasize the connection to quiver combinatorics.

1 Introduction

The central novel contribution of the article is the use of polytope mutations as a method for generating infinite staircases for symplectic ellipsoid embeddings. In particular, we give a uniform toric mutation explanation for the sharp embeddings in the two classical staircases: the McDuff-Schlenk’s Fibonacci Staircase [52, 53] and the Frenkel-Müller’s Pell Staircase [22], as well as the more recent discovery by Cristofaro-Gardiner-Kleinman [12]. In total, we present nine 4-dimensional symplectic toric domains \((X, \omega)\) with a sharp infinite staircase in its symplectic ellipsoid embedding function. These are conjecturally all such domains with this property, as formulated in [13, Conjecture 6.1]. This is the content of the first part of the article, presented in Sections 2 and 3.

The manuscript also develops a new understanding of symplectic tropical curves in almost-toric fibrations, incorporating the works of M. Symington [45, 65] and G. Mikhalkin [56, 57] into the study of symplectic ellipsoid embeddings, and the connections with the theory of cluster algebras and quiver mutations. This is the content of the second part of the article, developed in Sections 4 and 5.

1.1 Context and Results

Let \((\mathbb{R}^4, \omega_{st})\) be standard symplectic 4-space, \(a, b \in \mathbb{R}^+\), and consider the symplectic ellipsoid

\[ E(a_1, a_2) := \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : \frac{x_1^2 + y_1^2}{a_1} + \frac{x_2^2 + y_2^2}{a_2} \leq 1 \right\} \subseteq (\mathbb{R}^4, \omega_{st}). \]

Let \((X, \omega_X)\) be a 4-dimensional symplectic almost-toric domain, as introduced in Section 2. The goal of this article is to discuss the existence of infinite staircases for the function

\[ c_X(a) := \inf \{ \sigma \text{ such that } \exists i : (E(1, a), \omega_{st}) \hookrightarrow (X, \sigma \cdot \omega_{st}) \}, \]

where \(i\) denotes a symplectic embedding, i.e. \(i^* \omega_X = \omega_{st}\). A symplectic embedding is volume preserving, and thus the volume bound \(\pi^2 a \leq 2\sigma^2 \cdot \text{Vol}(X, \omega_{st})\) implies

\[ \frac{\pi \sqrt{a}}{\sqrt{2} \text{Vol}(X, \omega_{st})} \leq c_X(a). \]

The function \(c_X(a)\) is non-decreasing and continuous. The symplectic non-squeezing phenomenon [27] states the existence of values \(a \in \mathbb{R}^+\) for which the above inequality is actually strict. The ground-breaking work of D. McDuff and F. Schlenk [53] establishes for \((X, \omega_{st}) = (E(1, 1), \omega_{st})\) the
existence of a non-zero interval \( I_X = [\alpha_X, \Omega_X] \subseteq \mathbb{R}^+ \) and a convergent sequence \( S = \{s_n\}_{n \in \mathbb{N}} \subseteq I_X \) such that \( c_X(s_n) \) coincides with the volume lower bound and \( c_X|_{I_X \setminus S} \) is strictly larger than the volume bound. The exact graph for the function \( c_X \) in this case is depicted in [53, Figure 1.1].

**Definition 1.1** A 4-dimensional symplectic domain \((X, \omega_{st})\) is said to admit a sharp infinite staircase if there exists a non-zero interval \( I_X = [\alpha_X, \Omega_X] \subseteq \mathbb{R}^+ \) and an infinite sequence \( S = \{s_n\}_{n \in \mathbb{N}} \subseteq I_X \) of distinct points converging to \( \Omega_X \), such that \( c_X(s_n) \) coincides with the volume lower bound and \( c_X|_{I_X \setminus S} \) is strictly larger than the volume bound. The sharp points of a sharp infinite staircase \((I_X, S)\) are the points \( S = \{s_n\}_{n \in \mathbb{N}} \) where the volume bound for \( c_X|_{I_X} \) is sharp. □

The existence of a sharp infinite staircase for a 4-dimensional symplectic domain \((X, \omega_{st})\) has been a central question in the study of low-dimensional quantitative symplectic geometry [64]. In the first crucial discovery [53], it is shown that \((X, \omega_{st}) = (\mathbb{D}^4(1), \omega_{st})\), the standard unit ball, admits an infinite staircase. Two additional results were obtained for the polydisk \((X, \omega_{st}) = (\mathbb{D}^2(1) \times \mathbb{D}^2(1), \omega_{st} \oplus \omega_{st})\) by Frenkel-Müller [22], and for the ellipsoid \((X, \omega_{st}) = (\mathbb{E}(2, 3), \omega_{st})\) by Cristofaro-Gardiner-Kleinman [12]. See also the articles [7, 8, 32, 63, 66] and the enlightening survey [64] by F. Schlenk.

**Remark 1.2** Definition 1.1 is the notion we use in this manuscript, we refer to [66, Section 1.3] for comparison. The article [13] also considers non-sharp staircases where the volume lower-bound is not sharp. This corresponds to their \( J = 3 \) case, which would not abide by Definition 1.1. □

Let \( \mathcal{H} \) be the set of symplectic domains listed in Figure 1. The domains \((X, \omega_X) \in \mathcal{H}\) are obtained from one of the monotone closed symplectic 4-manifolds \( \mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, Bl_3(\mathbb{CP}^2) \) and \( Bl_4(\mathbb{CP}^2) \) by removing a configuration of surfaces. These surfaces depend on each item in Figure 1, and thus the choice of \((X, \omega_X) \in \mathcal{H}\). Each of these configurations consists of a union of symplectic 2-spheres, lying above the blue sides of the almost-toric base, and Lagrangian 2-spheres, located above the red cuts. These Lagrangian 2-spheres can be Hamiltonian isotoped to lie above the vertices where the cut starts, and thus \((X, \omega_X) \in \mathcal{H}\) can also be directly interpreted as the convex domain defined by the polytopes in Figure 1, without taking the red cuts into account.

First, let us state our main result on the construction of sharp ellipsoid embeddings:

**Theorem 1.3** Let \((X, \omega_X) \in \mathcal{H}\) be a 4-dimensional symplectic domain. Then the non-decreasing function \( c_X : \mathbb{R} \rightarrow \mathbb{R} \) admits a sequence of sharp points \( S \subseteq I_X \). □

Theorem 1.3 establishes the existence of sharp points, which is the central geometric ingredient to construct infinite staircases. Indeed, the existence of sharp infinite staircases consists of two independent arguments. First, a constructive result showing the existence of a sequence of sharp points \( S \subseteq I_X \). Second, an obstructive result stating that the volume bound is not an equality for the non-sharp points \( I_X \setminus S \). Our Theorem 1.3 strictly contributes with the constructive part, which is the part that the present manuscript geometrically establishes.

**Remark 1.4** The manuscript [13] uses the symplectic capacities from Embedded Contact Homology (ECH), as developed by M. Hutchings [35, 36], to provide the desired obstructions. The combination of our Theorem 1.3, being constructive, and the ECH obstructions in [13] imply that the sequence of sharp points \( S \subseteq I_X \) are indeed part of an infinite staircase. We refer to [13] for a detailed discussion and computation of these ECH capacities.
**Example 1.5** The three domains \((X, \omega_X) \in \mathcal{H}\) in the first row of Figure 1 are the unit ball \(\mathbb{D}^4(1)\), presented as the complement of a symplectic sphere \(\mathbb{CP}^1 \subseteq (\mathbb{CP}^2, \omega_{st})\), the polydisk \(\mathbb{D}^2(1) \times \mathbb{D}^2(1)\), arising as the complement of the two symplectic sphere \(\mathbb{CP}^1 \times \{pt\} \subseteq (\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{st} + \omega_{st})\), and \(E(1, 2)\), presented as the complement of a symplectic sphere \(\mathbb{CP}^1 \times \{pt\} \subseteq (\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{st} + \omega_{st})\) and a Lagrangian 2-sphere \(S^2 \subseteq (\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{st} + \omega_{st})\) in the homology class of the anti-diagonal. Theorem 1.3 for these three domains recovers the Fibonacci stairs [53] and Frenkel-Müller’s Pell stairs [22]. The Cristofaro-Gardiner-Kleinman staircase [13] for \(E(2, 3)\) correspond to the rightmost almost-toric base in the second row.

The symplectic domains \((X, \omega_X) \in \mathcal{H}\) not discussed in Example 1.5 do not have particular names, with the exception of the rightmost domain in the second row. For instance, the leftmost domain \((X, \omega_X) \in \mathcal{H}\) in the second row is the complement in \(\text{Bl}_3(\mathbb{CP}^2)\), \(\mathbb{CP}^2\) blown-up at three generic points \(p_1, p_2, p_3 \in \mathbb{CP}^2\), of a configuration of four symplectic 2-spheres: two of the exceptional divisors and the proper transforms of the projective line through \(p_1, p_2\), and the projective line through \(p_2, p_3\). These complements do not typically have a different named description, except for \(E(2, 3)\), which appears as the complement of an exceptional divisor and three Lagrangian 2-spheres in \(\text{Bl}_3(\mathbb{CP}^2)\).

![Diagram](image.png)

**Figure 1**: The almost-toric base diagrams for the symplectic domains \((X, \omega_X)\) in Theorem 1.3. The domains \((X, \omega_X)\) are the complements of symplectic 2-spheres, in blue, and Lagrangian 2-spheres, in red, in the closed almost-toric symplectic manifold corresponding to each diagram. Note that the same closed symplectic manifold might yield different symplectic domains \((X, \omega_X)\), as the choices of symplectic and Lagrangian 2-spheres depend on the almost-toric base diagram.

**Remark 1.6** We refer the reader to [13, Theorem 1.8] for a general statement on the precise value of the accumulation point for the ellipsoid embedding function \(c_X(a)\), should a sharp infinite staircase exist. Note that the result [13, Theorem 1.8] holds for any convex toric domain, and the accumulation points we obtain in our Theorem 1.3, as computed in Section 3, coincide exactly with that result. □
Second, this article presents a new and explicit method for showing the existence of sharp points by applying symplectic techniques coming from the combinatorics of almost-toric diagrams \[45, 65\], including toric mutations \([1, 2, 39]\), and the study of configurations of surfaces in symplectic almost-toric domains. In particular, in our proof for Theorem 1.3 we introduce symplectic-tropical curves in almost-toric fibrations, which represent smooth symplectic curves in 4-dimensional almost-toric symplectic manifolds. In a nutshell, our construction in Section 4 yields the following result:

**Theorem 1.7** Let \( \pi : (X, \omega_X) \to B \) be an almost-toric fibration and \( C \subseteq B \) a symplectic-tropical curve. Then there exists a symplectic curve \( C \subseteq X \) with \( \pi(C) = C \).

Tropical diagrams in symplectic almost-toric bases are introduced and developed in Section 4, which also proves Theorem 1.7. In fact, the tropical diagrams \( C \subseteq B \) in Theorem 1.7 allow us to readily compute homological intersection numbers and the construction of the infinite staircases in Theorem 1.7 relies on our study of the symplectic isotopy classes of the symplectic curves \( C_X \subseteq (\tilde{X}, \omega_{\tilde{X}}) \) associated to specific tropical diagrams \( C \). Theorem 1.7 further develops the work of M. Symington \[65\] in the direction of G. Mikhalkin’s study of complex tropical geometry \[56\].

Finally, the combinatorics and numerics appearing in the infinite staircases from Theorem 1.3 strongly intertwine with the recent developments in the study of cluster algebras \[18, 19\] and quiver mutations \([2, 39]\). This connection, and therefore the relation with homological mirror symmetry, is illustrated in Section 5.

**Organization.** The article is organized as follows. Section 2 introduces the basic ingredient in almost-toric symplectic geometry. Section 3 proves Theorem 1.3 by constructing the required infinite staircases with the results from Section 4. Section 4 proves Theorem 1.7, and Section 5 describes the combinatorics used for our infinite staircases in terms of quiver mutations.

**Acknowledgements.** We are grateful to Dan Cristofaro-Gardiner, Richard Hind, Liana Heuberger, Jeff Hicks, Tara Holm, Alessia Mandini, Dusa McDuff, Navid Nabijou, Ana Rita Pires, Laura Starkston and Wewei Wu for valuable discussions. R. Casals is supported by the NSF grant DMS-1841913, an Alfred P. Sloan Fellowship and a BBVA Research Fellowship. R. Vianna’s participation at the Matrix program on the Mirror Symmetry and Tropical Geometry at Creswick, Australia was significant for the development of technical aspects of symplectic-tropical curves in Section 4. R. Vianna is supported by Brazil’s National Council of scientific and technological development CNPq, via the research fellowships 405379/2018-8 and 306439/2018-2, and by the Serrapilheira Institute grant Serra-R-1811-25965.

**Relation to \[13\].** This article has been posted in parallel with the manuscript \[13\]. We are grateful to each of its authors, Dan Cristofaro-Gardiner, Tara Holm, Alessia Mandini and Ana Rita Pires, for the fluid and helpful communication with us. It is our understanding that both groups of authors came to the study of this problem from different perspectives, with the idea of using polytope mutations originating with the first author of the present manuscript. Both collaborations have benefited from our exchanges of ideas.

We encourage the reader to study the manuscript \[13\], which we find to be a very valuable contribution to the theory of symplectic ellipsoid embeddings as well. The results in our article, especially Theorem
1.3 are strengthen by their contributions to the *obstructive* side of the theory, as their manuscript [13] shows that ECH obstructions make the volume bound not sharp away from the required sequences. This clearly highlights the importance of the sharp ellipsoid embeddings we construct, and we gladly acknowledge the relevance and non-triviality of these ECH computations. Their manuscript [13] also proves Theorem 1.3, equally based on polytope mutations but concluding in a more succinct abstract manner, and formulates a compelling conjecture [13, Conjecture 1.18] regarding infinite staircases of rational convex toric domains. Each of the articles addresses the arithmetic of staircases from a different perspective: in this manuscript we directly use the Diophantine implicit equations, and the manuscript [13] proceeds parametrically, in terms of recursions. Even though both manuscripts could potentially be joined, we find that having both articles available is also enriching for the literature, as they discuss different techniques and perspectives.

\[\square\]

2 Preliminaries

2.1 Symplectic Almost-Toric Geometry

Let \( \mathcal{P} \) be an integral affine surface with singularities [28, Definition 1.24]. In this article, the regular part \( \mathcal{P}^{reg} \) is always presented as integrally embedded in the standard integral affine structure for \( \mathbb{R}^2 \).

For our purposes, it suffices for us to introduce the following definition.

**Definition 2.1** An almost-toric base \( \mathcal{P} = (P, \mathcal{B}) \) is a pair where

\[ P = \text{conv}(v_1, \ldots, v_{|V(P)|}) \subseteq \mathbb{R}^2 \]

is the polytope defined by the convex hull of \( |V(P)| \) vertices \( v_i \in \mathbb{R}^2, 1 \leq i \leq |V(P)| \) and \( \mathcal{B} \) is the cut content of \( \mathcal{P} \). The cut content \( \mathcal{B} \) is defined as the union of pairs \( B_i, 1 \leq i \leq |V(P)| \), where

\[ B_i = (c_i, n_i) \in \mathbb{R}^2 \times \mathbb{N} \]

consists of a vector \( c_i \in \mathbb{R} \) and a non-negative number \( n_i \in \mathbb{N} \).

The vector \( c_i \in \mathbb{R}^2 \) in the cut content indicates the direction of the cut at the vertex \( v_i \) and the number \( n_i \in \mathbb{N} \) is the number of singular nodes along the cut at \( v_i \). For instance, if \( v_i \) is a toric vertex then

\[ B_i = (\overline{0}, 0) \]

We have chosen the notation *cut content* in line with the *singular content* as defined in [39].

An almost-toric base \( \mathcal{P} = (P, \emptyset) \) with empty cut content yields a complex 2-dimensional toric variety \( X(P) \), as constructed in [10, 23]. In particular, \( (P, \emptyset) \) gives rise to a real 4-dimensional symplectic toric variety \( (X(P), \omega_{st}) \). The introduction of non-empty cut content \( \mathcal{B} \) geometrically corresponds to \( \mathbb{Q} \)-Gorenstein (partial) smoothing \( X(P, \mathcal{B}) \) of \( X(P) \) [39, 41]. Equivalently, there exists a degeneration of \( X(P, \mathcal{B}) \) into \( X(P) \).

**Example 2.2** Consider the five almost-toric bases \( \mathcal{P}_i = (P_i, \mathcal{B}_i) \), \( 1 \leq i \leq 5 \), depicted in Figure 2, increasingly ordered left to right. In this case, the five symplectic almost-toric varieties \( X(\mathcal{P}) \) are smooth 4-manifolds and, in fact, symplectic rescalings of \( (\mathbb{C}P^2, \omega_{st}) \). The toric varieties \( X(P_i) = X(P_i, \emptyset) \) are singular for \( 2 \leq i \leq 5 \), respectively corresponding to the weighted projective spaces \( \mathbb{C}P^2(1, 1, 4), \mathbb{C}P^2(1, 4, 25), \mathbb{C}P^2(1, 25, 169) \) and \( \mathbb{C}P^2(1, 169, 1156) \). This correspond to the fact that \( \mathbb{C}P^2 \) is a \( \mathbb{Q} \)-Gorenstein smoothing of these weighted projective spaces. \[\square\]
In Example 2.2, the addition of the cut contents $\mathcal{B}_i$ removes the singularities of the symplectic toric varieties $X(P_i, \emptyset)$, $2 \leq i \leq 5$. The singularities of the toric varieties we study in this article are cyclic quotient singularities, see [42, 70] and [39, Section 3.1]. In general, these singularities cannot be smoothed via a $\mathbb{Q}$-Gorenstein deformation, and a singular residue typically persists upon such a deformation [39]. The articles [15, 16] also discuss these singularities in the symplectic context.

The symplectic domains $(X, \omega_X) \in \mathcal{H}$ featured in Theorem 1.3 are open subsets of the symplectic 4-manifolds $X(P)$ obtained by removing collections of symplectic and Lagrangian 2-spheres. Let us introduce the combinatorial structure to encode these surface configurations.

**Definition 2.3** A relative almost-toric base consists of a pair $(\mathcal{P}, \mathcal{I})$ where $\mathcal{P} = (P, \mathcal{B})$ is an almost-toric base, and $\mathcal{I} = S \cup \mathcal{L}$ is a set consisting of sides of the polytope $P$, forming a subset $S \subseteq \mathcal{I}$, and segments $l \subseteq \mathcal{I}$ within each cut, a segment being bounded by two consecutive nodes, forming a subset $\mathcal{L} \subseteq \mathcal{I}$.

In the cases of Theorem 1.3, the sides $S \subseteq \mathcal{I}$ yield symplectic 2-spheres [45, 65] and the segments $\mathcal{L} \subseteq \mathcal{I}$ give rise to Lagrangian 2-spheres. Indeed, the cut content $\mathcal{B}$ of an almost-toric base defines a sequence of Lagrangian 2-spheres $L(B_i)$, $0 \leq j \leq n_i$, $1 \leq i \leq |V(P)|$. By definition, $L(B_i)$, $1 \leq j \leq n_i$, will be the exact Lagrangian 2-sphere with matching cycle the segment in the direction $c_i \in \mathbb{R}^2$ from the $j$th node to the $(j + 1)$th node, where nodes are ordered increasingly from the vertex $v_i$ outwards.

Consider a relative almost-toric base $(\mathcal{P}, \mathcal{I})$, we denote by $D(\mathcal{I}) \subseteq X(\mathcal{P})$ the configuration of symplectic divisors $D(S)$, associated to the pre-image of the sides of $S \subseteq \mathcal{I}$ under the almost-toric fibration, and by $L(\mathcal{L})$ the Lagrangian spheres associated to the segments in $\mathcal{L} \subseteq \mathcal{I}$. Note that for the same choice of sides $S$ in $P$, the topology of $D(S)$ typically depends on the cut content $\mathcal{B}$ of $\mathcal{P}$.

We shall denote

$$X(\mathcal{P}, \mathcal{I}) := X(\mathcal{P}) \setminus (D(S) \cup L(\mathcal{L})), $$

the symplectic complement of the symplectic divisors $D(S)$ and the Lagrangian spheres $L(\mathcal{L})$ in $X(\mathcal{P})$.

**Example 2.4** The unique cut in the rightmost relative almost-toric base $(\mathcal{P}, \mathcal{I})$ of the first row on Figure 1 yields a Lagrangian 2-sphere $S^2 \subseteq X(\mathcal{P}) \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. This is the unique Lagrangian sphere in $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the same Hamiltonian isotopy class as the anti-diagonal [33]. The blue side $S$ in this almost-toric base gives a symplectic divisor in the homology class of $D(S) = \mathbb{CP}^1 \times \{pt\} \subseteq X(\mathcal{P})$, and thus $X(\mathcal{P}, S) \cong \mathbb{CP}^1 \times D^2(1)$. Finally, the complement of the anti-diagonal Lagrangian and the symplectic 2-sphere $D(S)$ is $X(\mathcal{P}, \mathcal{I}) \cong E(1, 2)$.  


2.2 Polytope Mutations in Almost-Toric Diagrams

Let \((X, \omega_X)\) be a 4-dimensional closed toric symplectic variety \([10, 23]\). The moment map \(m : X \rightarrow \mathbb{R}^2\) associated to the Hamiltonian \(T^2\)-action yields a convex polytope \(m(X) = P_X \subseteq \mathbb{R}^2\). This polytope is the base \(\mathcal{P} = (P, \mathcal{B})\) of the Lagrangian toric fibration \(m : X \rightarrow m(X)\), which is a singular fibration at the boundary \(\partial(P_X) = m(X) \setminus \text{Int}(m(X))\), with isotropic fibers \([4, 5]\). The standard affine structure \(\mathbb{Z}^2 \subseteq \mathbb{R}^2\) in the real plane endows the base of this fibration with an affine structure, which itself coincides with the affine structure associated to the Lagrangian fibration \([3]\).

For \((X, \omega_X)\) an almost-toric symplectic variety, the analogue of the moment map \(m : X \rightarrow \mathbb{R}^2\) also yields a convex polytope \(m(X) = P_X \subseteq \mathbb{R}^2\) \([65]\). This polytope is the base of a singular Lagrangian toric fibration \(m : X \rightarrow m(X)\), which is a singular fibration also at finitely many points in the interior of \(P_X\). The standard affine structure \(\mathbb{Z}^2 \subseteq \mathbb{R}^2\) in the real plane restricted to the complement of the critical values of \(m\) yields a singular affine structure for the base. The choice of cuts for this singular affine structure yields an almost-toric base \(\mathcal{P} = (P, \mathcal{B})\).

Let \((X, \omega_X) = X(P, \mathcal{B})\) be an almost-toric symplectic variety with almost-toric base \((P, \mathcal{B})\), and \(v \in P\) be a vertex of its polytope \(P = m(X)\). Let us construct a family of Lagrangian almost-toric fibrations

\[
\mu_v^t(m) : (X, \omega) \rightarrow \mathbb{R}^2, \quad t \in [0, 1],
\]

with \(\mu_v^0(m) = m\) and the image \(\mu_v^1(m)(X)\) being the polytope \(\mu_v(P_X)\) obtained by combinatorially mutating \(P_X\) at \(v \in P_X\), as defined in \([1, \text{Section 3}]\). The crucial fact is that these are all fibrations of the same symplectic variety \((X, \omega_X)\). It is only the presentation of \((X, \omega_X)\) as an almost-toric symplectic domain that varies. In a nutshell, the geometric idea behind Theorem 1.3 is that different almost-toric fibrations for \((X, \omega_X)\) make different sharp embeddings of symplectic ellipsoids in \((X, \omega_X)\) visible.

Let us detail the three steps producing the family of fibrations \(\mu_v^t(m)\):

1. First, the introduction of a nodal trade at \(v\) and subsequent nodal slides, as defined by M. Symington \([65]\), produces a family of almost-toric fibrations \(\mu_v^t(m), \ t \in [0, 1/2]\), with \(\mu_v^0(m) = m\). In case that the cut content \(\mathcal{B}\) is non-zero at \(v\), the nodal trade is not necessary.

2. Second, a cut transfer \([65, 67]\). The almost-toric fibration \(\mu_v^{1/2}(m)\) admits a branch cut for its singular affine structure. This branch cut is defined as the unique straight segment connecting the image of the unique nodal fiber of \(\mu_v^{1/2}(m)\) with the vertex \(v \in P_X\). In the complement of this segment \(s_v \subseteq P_X\), the fibration \(\mu_v^{1/2}(m)\) is trivial. The almost-toric fibration \(\mu_v^{1/2}(m)\) can now be described differently, by performing a nodal transfer introducing a, potentially new, vertex \(\tilde{v}\).

Indeed, consider the two piece partition \(P_X = P_X^{(1)} \cup P_X^{(2)}\) given by the line \(L_v\) containing \(s_v\) and apply a \(\text{SL}(2, \mathbb{Z})\) transformation \(A\) to \(P_X^{(2)}\) which preserves \(s_v\) and turns the vertex \(v \in P_X\) into an inner point of the unique straight line containing the unique side of \(P_X^{(1)}\) which contains \(v\). This yields a new polytope \(\mu_v(P_X) = P_X^{(1)} \cup A(P_X^{(2)})\), with a new vertex \(\tilde{v} = (L_v \cap \partial P_X) \setminus \{v\}\) and a new Lagrangian almost-toric fibration \(\mu_v^{1/2}(m)\) with a unique nodal fiber.

3. Third, by performing nodal slides and a nodal trade to \(\tilde{\mu}_v^{1/2}(m)\) along the unique segment connecting the image of the nodal fiber and the new vertex \(\tilde{v}\), we obtain a family \(\tilde{\mu}_v^t(m), \ t \in [1/2, 1]\), with \(\tilde{\mu}_v^1(m)\) an almost-toric fibration, whose underlying toric moment polytope is the combinatorial mutation \(\mu_v(P_X)\). Given the identification \(\mu_v^{1/2}(m) = \tilde{\mu}_v^{1/2}(m)\), we denote the concatenation of \(\mu_v^t(m), \ t \in [0, 1/2]\), and \(\tilde{\mu}_v^t(m), \ t \in [1/2, 1]\) by \(\mu_v^t(m), \ t \in [0, 1]\).
The family of almost-toric fibrations \( \{ \mu^t_n(\mathbf{m}) \}_{t \in [0,1]} \), as constructed above, has the property that for small enough \( t \in [0,1] \), the nodal fiber of the almost-toric fibration \( \mu^t_n(\mathbf{m}) \) is arbitrarily close to the new node of \( \mu_n(P_X) \). For now, let us assume that \( P_X \) is a triangle, which will simplify our notation.

The process described in the three steps above can be iterated for the initial symplectic domain \((X, \omega_X)\), as follows. Let \( V = (v_n)_{n \in \mathbb{N}} \) be a sequence of vertices, with \( v_1, v_2 \) distinct, such that

\[ v_i \in \mu_{v_{i-1}} \mu_{v_{i-2}} \circ \ldots \circ \mu_{v_1}(P_X), \quad i \in \mathbb{N}, \]

and \( v_j = \tilde{v}_{j-2} \) for all \( j \in \mathbb{N} \). In short, we choose two vertices \( v_1, v_2 \in P_X \) in a triangle and we first mutate at \( v_1, v_2 \). Then we mutate at the new vertex \( v_3 = \tilde{v}_1 \in \mu_{v_2} \mu_{v_1}(P_X) \) which was first opposite to \( v_1 \) in \( P_X \), and then at \( v_4 = \tilde{v}_2 \in \mu_{v_3} \mu_{v_2}(P_X) \), which was initially opposite to \( v_2 \) in \( \mu_{v_1}(P_X) \). Note that the triangle \( P_X \) has a vertex \( v_j \in P_X \) which remains fixed under these iterative mutations.

**Example 2.5** Figure 2 shows a sequence of rescaled almost-toric fibrations \( \{ \mu^t_n(\mathbf{m}) \}_{t \in [0,4]} \) at the values \( t = 0, 1, 2, 3, 4, \) on the symplectic manifold \((\mathbb{C}P^2, \omega_{\text{std}})\). First, there is a mutation from the toric moment polytope \( P \) for \((\mathbb{C}P^2, \omega_{\text{std}})\) to \( \mathcal{P}_1 = (P_1, \mathcal{B}_1) \), where \( \mu_1(P) \) is the toric moment polytope for \( \mathbb{C}P^2(1,1,4) \). The second mutations moves from \( \mathcal{P}_1 = (\mu_1(P), \mathcal{B}_1) \) to \( \mathcal{P}_2 = (\mu_2(P), \mathcal{B}_2) \), where \( \mu_2(P) \) is the toric moment polytope for \( \mathbb{C}P^2(1,4,25) \). The third mutation leads to the almost-toric base with polytope \( \mu^3(P) \), the toric moment polytope for \( \mathbb{C}P^2(1,25,169) \), and the fourth mutation arrives at \( \mu^4(P) \), the toric moment polytope for \( \mathbb{C}P^2(1,169,1156) \). This procedure can be iterated indefinitely, yielding almost-toric bases with their polytope \( \mu^t(P) \) being the toric moment polytope for \( \mathbb{C}P^2(1,a_n, b_n) \), where \((a_n, b_n)\) will always be squares of consecutive odd Fibonacci numbers. A clear understanding of this sequence of almost-toric mutations makes Theorem 1.3 for the unit 4-ball \( \mathbb{D}^4(1) \) much more intuitive. \( \square \)

### 3 Existence of Infinite Staircases

The central idea that this manuscript introduces is the use of *almost-toric mutations* in the study of symplectic ellipsoid embeddings. In this section we prove our main result Theorem 1.3, assuming Theorem 3.12, which expands Theorem 1.7, and whose proof will require the tropical symplectic techniques developed in Section 4.

Let us start by explaining how to construct symplectic ellipsoid embeddings in an almost-toric domain \( X(\mathcal{P}, \mathcal{S}) \) with \((\mathcal{P}, \mathcal{S})\) a relative almost-toric base.

**Definition 3.1** Let \( P \subseteq \mathbb{R}^2 \) be a convex polygon and \( v_0, v_1, v_2 \in V(P) \) three vertices of \( P \). The triangle \( \overline{T}_{v_0,v_1,v_2} \subseteq P \) is the convex hull of \( v_0, v_1, v_2 \). For \( \varepsilon \in \mathbb{R}_{>0} \), the open \( \varepsilon \)-triangle \( \overline{T}_{v_0,v_1,v_2}^{\varepsilon} \subseteq P \) is the complement \( \overline{T}_{v_0,v_1,v_2} \setminus \varepsilon^\circ \), where \( \varepsilon^\circ \subseteq \overline{T}_{v_0,v_1,v_2} \) is an \( \varepsilon \)-neighborhood of the (open) side \( \text{conv}(v_0, v_2) \)

Let \((X, \omega) = X(\mathcal{P}, \mathcal{S})\) be symplectic almost-toric manifold with polytope \( P \subseteq \mathcal{P} \), and assume that \( v_1 \in V(P) \) is a smooth toric vertex. Let \( v_0, v_2 \) be the two vertices in \( P \) closest to \( v_1 \). If the open triangle \( T_{v_0,v_1,v_2}^\varepsilon \subseteq P \) does not contain any critical values for the almost-toric fibration \( m_X: X \rightarrow P \), the pre-image of \( T_{v_0,v_1,v_2}^\varepsilon \) under \( m \) yields a symplectic embedding

\[ i_{v_1} : E(a, b) \rightarrow (X, \omega), \]
where $a, b$ are the affine lengths of the two sides $\text{conv}(v_0, v_1) \cap T^e_{i_0, v_1, v_2}$, $\text{conv}(v_1, v_2) \cap T^e_{i_0, v_1, v_2}$ respectively. This construction is a potential method for construction symplectic ellipsoid embeddings, but it has the following two disadvantages.

First, for a given $\varepsilon \in \mathbb{R}^+$, there are only finitely many open triangles in a moment polytope $P \subseteq \mathbb{R}^2$ for $(X, \omega)$ and thus, even if we considered the $GL(2, \mathbb{Z})$ action, this observation on its own is not sufficient to build an infinite sequence of ellipsoid embeddings. Second, unless $P$ is a triangle, i.e. $|V(P)| = 3$, the symplectic ellipsoid embeddings of the form $i_{v_1}$ will not be volume-filling.

The first new geometric idea is that the symplectic manifold $(X, \omega)$ admits a sequence of almost-toric fibrations $(m_n)_{n \in \mathbb{N}} : X \to \mathbb{R}^2$, as introduced in Section 2.2, which themselves can be used to construct a sequence of symplectic ellipsoid embeddings. The images $m_n(X)$ of these almost-toric fibrations are also convex polytopes $P_n \subseteq \mathbb{R}^2$. In this almost-toric case, the symplectic embeddings $i_{v_1}$ are built as above, where the main condition is that the open triangles $T^e_{\varepsilon_{i_0}, v_1, v_2} \subseteq P$ do not include any interior singular values of $m_n$. (This condition is always satisfied in the toric case, as there are no interior singular values for the Lagrangian fibration.)

In Subsection 3.1, we describe the properties for a sequence of almost-toric fibrations $(m_n)_{n \in \mathbb{N}} : X \to \mathbb{R}^2$ such that each almost-toric fibration admits a volume-filling open triangle $T^e_{\varepsilon_{i_0}, v_1, v_2} \subseteq P$, for arbitrarily small $\varepsilon \in \mathbb{R}^+$. These open triangles give rise to symplectic embeddings

$$i^{(n)}_{v_1} : E(a_n, b_n) \to (X, \omega).$$

The arithmetic of the sequence of pairs $(a_n, b_n) \in \mathbb{N}^2$ is governed by a Diophantine equation which depends on the initial choice of $(\mathcal{P}, \mathcal{J})$. These Diophantine equations have featured prominently in birational geometry [30, 37, 38] and the study of coherent sheaves [26, 61, 62], and we present them systematically in Subsection 3.2.

**Remark 3.2** The use of almost-toric fibrations has many advantages, including the fact that we can build sequences of almost-toric fibrations using the theory of mutations in algebraic geometry [1, 2, 30]. Nevertheless, the standard techniques have the disadvantage that it requires our symplectic almost-toric manifold $(X, \omega)$ to be closed. That said, the known tools for polytope mutations and algebraic degenerations do not include the relative case of symplectic divisors and Lagrangian submanifolds. This article starts developing techniques in this direction.

In Theorem 1.3, we are interested in open symplectic toric domains $X(\mathcal{P}, \mathcal{J})$, with $\mathcal{J} \neq \emptyset$. In order to address this dissonance, we first compactify $X(\mathcal{P}, \mathcal{J})$ to the closed symplectic toric manifold $X(\mathcal{P})$ by adding the surface configuration $D(\mathcal{P}) \subseteq (X, \omega)$. Then we construct the sequence of almost-toric fibrations $(m_n)_{n \in \mathbb{N}} : X(\mathcal{P}) \to \mathbb{R}^2$ and, at the same time, keep track of the images of $D(\mathcal{P}) \subseteq (X, \omega)$ under these almost-toric fibrations. We achieve this latter step by developing a new theory of tropical symplectic curves, in Section 4, and use the uniqueness of certain symplectic isotopy classes in symplectic 4-manifolds [43, 50, 51] and the classification of Lagrangian 2-sphere in the monotone symplectic 4-manifolds $\mathbb{C}P^1 \times \mathbb{C}P^1$, $Bl_3(\mathbb{C}P^2)$, $Bl_4(\mathbb{C}P^2)$ [6, 33, 46, 47].

In conclusion, the two main ingredients to execute the above scheme, and thus prove Theorem 1.3, are:

1. The construction of a sequence $(m_n)_{n \in \mathbb{N}} : X \to \mathbb{R}^2$ of almost-toric fibrations, and its associated volume-filling symplectic ellipsoid embeddings $i^{(n)} : E(a_n, b_n) \to (X, \omega)$. This is the content of Subsections 3.1 and 3.2.
(ii) Showing that the images $t^{(n)}(E(a_n, b_n))$ of the symplectic embeddings in Part (i) lie in the complement of the compactifying divisor $D(\mathcal{D}) \subseteq X(\mathcal{P}, \mathcal{I})$. This will be argued in Subsection 3.3.

The proof of Theorem 1.3 then proceeds in the following three steps. First, the existence and choice of a Symington Sequence, which addresses the first item above. Second, the construction of tropical configurations realizing $D(\mathcal{D})$ at each stage of the Symington Sequence, in Section 4. Third, the study of the Hamiltonian isotopy class of these configurations in $X(\mathcal{P})$, which is the content of Subsection 3.3. The second and third steps address the second item above.

### 3.0.1 The symplectic domains in Figure 1

The combinatorial reason that sharp ellipsoid staircases can be constructed for the domains in Figure 1 is that they have almost-toric basis with a smooth toric vertex, which admit infinitely many polytope mutations to triangular almost-toric polytopes, with the smooth toric vertex fixed. The nine polytopes in Figure 1 are a subset of the 16 reflexive polytope [39]. These reflexive polytopes also yield one symplectic domains inside of $Bl_1(\mathbb{C}P^2)$, and two symplectic domains inside of $Bl_2(\mathbb{C}P^2)$, the three of them with infinitely many triangular polytope mutations. The same arguments presented in this article construct infinitely many ellipsoid embeddings $E(1, a_n)$, with $(a_n)_{n \in \mathbb{N}}$ a convergent sequence. Nevertheless, these ellipsoid embeddings are not sharp. The twelve symplectic domains presented in [12] are precisely the nine domains in Figure 1 with the addition of these three domains. At this stage, it seems rather natural to ask about the remaining four, of the sixteen, reflexive polytopes. It might be possible to extract interesting staircases using our polytope mutation methods, however, the sequence of polytope mutations shall not be of triangular polytopes. The reader is referred to [1, 39] for a thorough study of the mutation classes of reflexive polytopes.

### 3.1 Realization of a Symington Sequence

Given an almost-toric symplectic 4-manifold $X(\mathcal{P})$, it is possible to mutate an almost-toric fibration for $X(\mathcal{P})$ as in Subsection 2.2 in many ways. It is not true that any mutation, even if the underlying polytope is always triangular, will yield a sharp symplectic ellipsoid embedding.

**Example 3.3** We can mutate the triangular polytope $P_{\mathbb{C}P^2}$ for $\mathbb{C}P^2 = X(P_{\mathbb{C}P^2}, \emptyset)$ to the toric moment polytope for the weighted projective space $\mathbb{C}P^2(2, 5, 29)$. In fact, there are infinitely many $\mathbb{Q}$-Gorenstein degenerations of $\mathbb{C}P^2$ obtained by mutating the triangular polytope $P_{\mathbb{C}P^2}$, including $\mathbb{C}P^2(5, 29, 433)$ and $\mathbb{C}P^2(29, 169)$ [2, 30]. Neither of these weighted projective spaces readily admits a sharp symplectic ellipsoid embedding, since the three vertices of their toric moment polytopes are singular. 

Example 3.3 illustrates that a specific choice of sequence of mutations is needed in order to construct sharp ellipsoid embeddings with our method. This leads to the following definition, which we have named after M. Symington, after her exemplary article [65].

**Definition 3.4** A *Symington sequence* for an almost-toric symplectic manifold $X(\mathcal{P}, \mathcal{I})$ consists of a sequence of pairs $\{(\mathcal{P}_n, v_n)\}_{n \in \mathbb{N}}$ such that:

1. $X(\mathcal{P}_n) = X(\mathcal{P})$ for all $n \in \mathbb{N}$,
(S2) $P_n$ is a triangle and $v_n \in V(P_n)$ is a vertex,
(S3) $P_{n+1} = \mu_{v_n}(P_n)$, where $\mu_{v_n}(P_n)$ is the polytope mutation of $P_n$ at $v_n$,
(S4) There exists $v_f \in \mathbb{R}^2$ such that $v_f \in V(P_n)$ is a smooth toric vertex, for all $n \in \mathbb{N}$.

The vertex $v_f$ in (S4) will be referred to as a frozen vertex, as it does not appear in the sequence $(v_n)_{n \in \mathbb{N}}$ specifying the sequence of mutations $(\mu_{v_n})_{n \in \mathbb{N}}$.

The crucial geometric properties in Definition 3.4 are (S1) and (S3), and the fact that we are interested in ellipsoid embeddings leads to requiring (S2) and (S4).

**Proposition 3.5** Let $X(\mathcal{P}, \mathcal{I})$ be a symplectic domain with relative almost-toric base $(\mathcal{P}, \mathcal{I}) \in \mathcal{H}$. Then there exists a Symington sequence $X(\mathcal{P}, \mathcal{I})$.

**Proof** The statement readily holds for those $X(\mathcal{P}, \mathcal{I})$ such that $P \in \mathcal{P}$ is a triangle with a smooth toric vertex. Indeed, if $P$ is a triangle, a Symington sequence is obtained by choosing one of the two vertices in $P$ different from $v_f$, and iteratively mutating and choosing the only vertex in the mutated polytope which differs from $v_f$ and the newly created vertex. In case $P \in \mathcal{P}$ is not a triangle, we must directly verify that the almost-toric diagram $\mathcal{P}$ can be mutated to triangular almost-toric diagram with a smooth toric vertex. This is explicitly shown in [69], and Figure 1 for $Bl_3(\mathbb{CP}^2)$. 

The arithmetic of the vertices of a Symington sequence for each $X(\mathcal{P}, \mathcal{I}) \in \mathcal{H}$ is discussed in the subsequent Subsection 3.2. The usefulness of a Symington sequence in the study of the Ellipsoid Embedding function $c_X$ comes from the following:

**Proposition 3.6** Let $P \subseteq \mathbb{R}^2$ be a triangular polytope with a smooth toric vertex $v_f \in V(P)$, and let $(a, b) \in \mathbb{Z}^2$ be the affine length of the two sides of $P$ incident to $v_f$. Then there exists a volume-filling symplectic embedding $i : E(a, b) \longrightarrow X(P)$. 

Proposition 3.6 follows by noticing that the complement of the symplectic divisor associated to the side in $P$ non-incident to $v_f$ is the symplectic domain $E(a, b)$ [23, 34, 64]. In particular, let $\{(\mathcal{P}_n, v_n)\}_{n \in \mathbb{N}}$ be a Symington sequence and denote by $(a_n, b_n) \in \mathbb{Z}^2$ the affine lengths of the sides of $P_n$ incident to $v_f$. Then there exist a sequence of symplectic ellipsoid embeddings $i_n : E(a_n, b_n) \longrightarrow X(P_n)$.

This is at the core of the relation between the existence of infinite staircase and polytope mutations. It is a powerful starting point, but it not enough to conclude Theorem 1.3 since the symplectic varieties $X(P_n)$ are not isomorphic. As emphasized, $X(P_n)$ are typically singular algebraic varieties, and their algebraic isomorphic type strongly depends on $n \in \mathbb{N}$. That said, the fundamental defining property (S1) of a Symington sequence states that $X(\mathcal{P}_n) = X(\mathcal{P})$ for all $n \in \mathbb{N}$.

The difference between the equalities $X(\mathcal{P}_n) = X(\mathcal{P})$, for all $n \in \mathbb{N}$, and the sequence of algebraic varieties $X(P_n)$ is contained in the cut content $\mathcal{P}_n$ in $\mathcal{P}_n = (P_n, \mathcal{P}_n)$. In particular, Proposition 3.6 implies the following:

**Proposition 3.7** Let $P \subseteq \mathbb{R}^2$ be a triangular polytope with a smooth toric vertex $v_f \in V(P)$, and let $(a, b) \in \mathbb{Z}^2$ be the affine length of the two sides of $P$ incident to $v_f$. Suppose that $\mathcal{P} = (P, \mathcal{P})$ has empty cut content at $v_f$. Then there exists a volume-filling symplectic embedding $i : E(a, b) \longrightarrow X(\mathcal{P})$. 

\[ \square \]
Indeed, introducing cut content $\mathcal{B}$ can be achieved by choosing arbitrarily short cuts at the vertices of $P$. In precise terms, let $N \subseteq P$ be a neighborhood of the side in $P$ opposite to $v_f$ containing the cuts in $\mathcal{B}$. The complement in $X(\mathcal{P})$ of the pre-image of $N \subseteq P$ is a symplectic ellipsoid $E(a - \varepsilon, b - \varepsilon)$ for certain $\varepsilon \in \mathbb{R}^+$ depending on the choice of $N$. By choosing $N \subseteq \mathbb{R}^2$ small enough in Euclidean area, $\varepsilon \in \mathbb{R}^+$ can be chosen arbitrarily small, thus yielding a volume-filling embedding of $E(a, b)$.

Given a Symington sequence $\{(\mathcal{P}_n, v_n)\}_{n \in \mathbb{N}}$ for $X(\mathcal{P}, \mathcal{I})$, Proposition 3.5 and Proposition 3.7 allow us to construct a sequence of sharp symplectic ellipsoid embeddings

$$i_n : E(a_n, b_n) \rightarrow X(\mathcal{P}).$$

The challenge is now upgrading these absolute ellipsoid embeddings $(i_n)_{n \in \mathbb{N}}$ to a sequence of relative sharp symplectic ellipsoid embeddings

$$i_n : E(a_n, b_n) \rightarrow X(\mathcal{P}, \mathcal{I}).$$

This would conclude Theorem 1.3 once we study the arithmetic properties of the Symington sequence $(a_n, b_n)$ associated to $X(\mathcal{P}, \mathcal{I})$ as in Proposition 3.5. The focus of Subsection 3.3 and Section 4 is the construction of the relative embeddings $i_n$ into $X(\mathcal{P}, \mathcal{I})$ from the absolute embeddings $i_n$ into $X(\mathcal{P})$.

The central difficulty in arguing the existence of $i_n$ is understanding the surface configuration $D(\mathcal{I}) \subseteq X(\mathcal{P})$ under the identifications $X(\mathcal{P}) \cong X(\mathcal{P}_n)$, for each $n \in \mathbb{N}$. In order to achieve that, we will develop a diagrammatic tropical calculus for surface configurations $D(\mathcal{I}) \subseteq X(\mathcal{P})$ in almost-toric symplectic manifolds. In particular, such tropical calculus starts with a configuration $D(\mathcal{I}) \subseteq X(\mathcal{P}_n)$ and describes a diagram $\Delta_n \subseteq \mathcal{P}_n$ such that the Hamiltonian isotopy class of the configuration $D(\mathcal{I})$ admits a representative contained in the pre-image of this diagram $\Delta_n$.

**Remark 3.8** There is an alternative course of action to construct $i_n$. It should be possible to understand how the surface configuration $D(\mathcal{I})$ is explicitly carried along a polytope mutation through almost-toric diagrams [65, 69]. Starting with $D(\mathcal{I}) \subseteq X(\mathcal{P})$, this would yield an understanding of the inclusions $D(\mathcal{I}) \subseteq X(\mathcal{P}_n)$, upon identifying $X(\mathcal{P}_n) \cong X(\mathcal{P})$ along the mutation sequence. This is an interesting line of research, but we shall not discuss it in the present manuscript.

Let us now provide a detailed and self-contained account of the numerics appearing in Symington sequences for $X(\mathcal{P}, \mathcal{I})$ for $(\mathcal{P}, \mathcal{I}) \in \mathcal{H}$. After that, we proceed with Subsection 2.1, which states the main result from Section 4, and Subsection 3.3, which constructs $(i_n)_{n \in \mathbb{N}}$, leading to the conclusion of Theorem 1.3.

### 3.2 Arithmetic of Symington Sequences for $(\mathcal{P}, \mathcal{I}) \in \mathcal{H}$

The polytope $P \in \mathcal{P}$ in an almost-toric base contains all the arithmetic information for its mutations. The polytopes in Figure 1 are mutation equivalent to a triangular polytope. This allows us to reduce the arithmetic of Symington sequences for the polytopes in Figure 1 to those for the weighted projective spaces $\mathbb{C}P^2(1, 1, 1), \mathbb{C}P^2(1, 1, 2), \mathbb{C}P^2(1, 2, 3)$ and $\mathbb{C}P^2(1, 4, 5)$. The following result covers all the necessary arithmetic for our Theorem 1.3:

**Proposition 3.9** ([2, 69]) Let $P(\alpha, \beta, \gamma)$ the toric moment polytope for $\mathbb{C}P^2(\alpha, \beta, \gamma)$.

(i) Suppose that $\alpha$ corresponds to a smoothable singularity of $\mathbb{C}P^2(\alpha, \beta, \gamma)$ and divides $(\beta + \gamma)^2$, with $\alpha \cdot \delta = (\beta + \gamma)^2$. Then the polytope $P(\beta, \gamma)$ admits a polytope mutation to the triangle $P(\beta, \gamma, \delta)$.  

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(ii) \textit{(Vieta jumping)} Suppose that \((a, b, c) \in \mathbb{Z}^3\) solves the Diophantine equation

\[ C_0a^2 + C_1b^2 + C_2c^2 = mabc. \]

Suppose that the \(C_i\) divide \(m\), \(1 \leq i \leq 3\), then either of the three triples \((a, b, mab/C_2 - c)\), \((a, mac/C_1 - b, c)\) and \((mac/C_0 - a, b, c)\) solves this Diophantine equation.

\[ \square \]

\textbf{Remark 3.10} A toric orbifold singularity being smoothable is equivalent to being a \(T\)-singularity \([2, 39]\). If \(\mathbb{CP}^2(\alpha, \beta, \gamma)\) is smoothable, then it is of the form \(\mathbb{CP}^2(C_0a^2, C_1b^2, C_2c^2)\), where \((a, b, c) \in \mathbb{Z}^3\) solves the Diophantine equation above. Vieta jumping is then equivalent to the change \(\alpha \leftrightarrow \delta\) in Proposition 3.9(i), with \(\delta = C_0(mac/C_0 - a)\). Note that Proposition 3.9 is proven in \([2]\) for more general toric orbifolds with triangular moment map and smoothable corner, and it appears in the context of Del Pezzo surfaces in \([69]\) as Lemma 4.2.

\[ \square \]

\textbf{Arithmetic for} \((\mathbb{CP}^2, \omega_{\text{st}})\). The toric moment polytope for \((\mathbb{CP}^2, \omega_{\text{st}})\) is mutation equivalent to the toric moment polytope for \(\mathbb{CP}^2(a^2, b^2, c^2)\) if and only if \((a, b, c) \in \mathbb{Z}\) is a Markov triple, i.e.

\[ a^2 + b^2 + c^2 = 3abc. \]

This explains the numerics in Example 3.3. An arbitrary sequence of Markov triples, even if they differ only in one component, does not yield a Symington Sequence, since condition (S4) in Definition 3.4 imposes a non-trivial constraint. Indeed, \(\mathbb{CP}^2(a^2, b^2, c^2)\) has a smooth toric vertex if and only if one of the numbers \(a, b, c \in \mathbb{Z}\) equals one. Thus a Symington Sequence for \((\mathbb{CP}^2, \omega_{\text{st}})\) can be obtained using the construction in the proof of Proposition 3.5 yielding any of the toric polytopes for \(\mathbb{CP}^2(1, b^2, c^2)\), if \((b, c) \in \mathbb{Z}\) satisfy

\[ 1 + b^2 + c^2 = 3bc. \]

Let \(F_n\) be the \(n\)th odd Fibonacci number, starting at \(F_1 = 1\). The recursion \(F_{n+2} = 3F_{n+1} - F_n\) implies that \((1, F_n, F_{n+1})\) is a Markov triple with \(a = 1\), as required. Hence, a Symington Sequence for \((\mathbb{CP}^2, \omega_{\text{st}})\) yields the sequence of toric polytopes \(\mathbb{CP}^2(1, F_n^2, F_{n+1}^2)\) for all \(n \in \mathbb{N}\). Conversely, all positive integral solutions \((1, b, c)\) of the Markov equation are of the form \((1, F_n, F_{n+1})\). These Fibonacci solutions are directly obtained by iteratively applying the Vieta jumping in Proposition 3.9 starting with the minimal solution \((a, b, c) = (1, 1, 1)\).

This is our explanation for the appearance of the odd Fibonacci numbers in D. McDuff’s Fibonacci staircase from the viewpoint of polytope mutations. Indeed, this computation shows the \(E(1, F_{n+1}^2/F_n^2)\) admits a sharp symplectic embedding into

\[ \left( \mathbb{CP}^2, \sqrt{\frac{2 \text{Vol}(E(1, F_{n+1}^2/F_n^2))}{\pi^2}} \cdot \omega_{\text{st}} \right), \]

and that any symplectic ellipsoid embedding obtained via toric mutations must be of this form.

\textbf{Remark 3.11} From the proof of Theorem 6.5 in \([44]\), see \([44, \text{Figure 15}]\), we can readily conclude that, if \(F_{n+1} > F_n\), then \(E(1, F_{n+1}^2/F_n^2)\) embeds into

\[ \left( B_k(\mathbb{CP}^2), \frac{F_{n+1}^2/F_n^2}{3F_{n+1}/F_n - k} \sqrt{\frac{2 \text{Vol}(E(1, F_{n+1}^2/F_n^2))}{\pi^2}} \cdot \omega_{\text{st}} \right), \quad k \leq 5. \]
Similarly, the ellipsoid $E(1, F_n^2/F_{n+1}^2)$ embeds into
\[
\left( Bl_k(\mathbb{CP}^2), \frac{F_n^2/F_{n+1}^2}{3F_n/F_{n+1} - k} \sqrt{\frac{2 \text{Vol}(E(1, F_n^2/F_{n+1}^2))}{\pi^2}}, \omega_{\text{st}} \right),
\]
if $F_n > F_{n+1}$ and $k \leq 5$. The cases $k = 1$ and $2$ are essentially depicted in Figures 7 and 8 of [69], up to one mutation on the rightmost diagram of Figure 8. We learned about staircases in the toric domains related to $Bl_1(\mathbb{CP}^2)$ and $Bl_2(\mathbb{CP}^2)$ from [13].

**Arithmetic for $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{\text{st}} \oplus \omega_{\text{st}})$**. The toric moment polytope for $(\mathbb{CP}^2(1, 1, 2), \omega_{\text{st}})$ is mutation equivalent to the toric moment polytope for $\mathbb{CP}^2(a^2, b^2, 2c^2)$ if and only if $(a, b, c) \in \mathbb{Z}$ satisfies
\[
a^2 + b^2 + 2c^2 = 4abc.
\]
As above, these $\mathbb{CP}^2(a^2, b^2, 2c^2)$ have smooth toric vertex if and only if one of the numbers $a, b, c \in \mathbb{Z}$ equals one. Thus a Symington Sequence for $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{\text{st}} \oplus \omega_{\text{st}})$ exists and yields any of the toric polytopes for $\mathbb{CP}^2(1, b^2, 2c^2)$, if $(b, c) \in \mathbb{Z}$ satisfy
\[
1 + b^2 + 2c^2 = 4bc.
\]
Let $P_n, H_n$ be the $n$th Pell number and the $n$th half-companion Pell number, starting at
\[
P_0 = 1, P_1 = 1, \quad H_0 = H_1 = 1.
\]
It is readily verified that the triples $(1, 2P_n, H_n^2)$, verify the above Diophantine equation, since $P_n = 2P_{n-1} + P_{n-2}$ and $H_n = 2H_{n-1} + H_{n-2}$. These Pell solutions are also obtained by iteratively applying Proposition 3.9 starting with $(a, b, c) = (1, 1, 1)$.

Consider $c_n = \max \{2P_n^2/H_n^2 + 1, H_n^2/(2P_n^2) \}$, this computation shows the symplectic ellipsoid $E(1, c_n)$ symplectically embeds into $(\mathbb{CP}^1 \times \mathbb{CP}^1, \text{Vol}(E(1, c_n)) \cdot (\omega_{\text{st}} \oplus \omega_{\text{st}}))$ for all $n \in \mathbb{N}$, where $\text{Vol}$ denotes the rescaled volume, as in Remark 3.11. This explains the numerics in Frenkel-Müller’s Pell staircase via polytope mutations.

**Arithmetic for $(Bl_3(\mathbb{CP}^2), \omega_{\text{st}})$**. The toric moment polytope for $(\mathbb{CP}^2(1, 2, 3), \omega_{\text{st}})$ is mutation equivalent to the toric moment polytope for $\mathbb{CP}^2(a^2, 2b^2, 3c^2)$ if and only if $(a, b, c) \in \mathbb{Z}$ satisfies
\[
a + 2b^2 + 3c^2 = 6abc.
\]
A Symington Sequence for $(Bl_3(\mathbb{CP}^2), \omega_{\text{st}})$ thus yields any of the toric polytopes for $\mathbb{CP}^2(1, b, c)$, if $(b, c) \in \mathbb{Z}$ satisfy
\[
1 + 2b^2 + 3c^2 = 6bc.
\]
This Diophantine equation gives rise to the numerics of the Cristofaro-Gardner-Kleinman’s staircase. Indeed, Vieta jumping alternately applied to the second and third components of $(a, b, c) = (1, 1, 1)$ yields the sequence of triples
\[
(1, 1, 1) \mapsto (1, 2, 1) \mapsto (1, 2, 3) \mapsto (1, 7, 3) \mapsto \ldots
\]
which prove the existence of sharp ellipsoid embeddings into the appropriately rescaled $(Bl_3(\mathbb{CP}^2), \omega_{\text{st}})$, starting with the corresponding sequence
\[
E(1, 3/2), E(1, 8/3), E(1, 27/8), E(1, 98/27), \ldots,
\]
associated to the triples above. The sequence $\{k_n\}_{n \in \mathbb{N}}$ of ratios of solutions for the Diophantine equation obtained by Proposition 3.9 is a convergent infinite sequence, limiting to $2 + \sqrt{3}$. 

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Arithmetic for \((\text{Bl}_d(\mathbb{CP}^2), \omega_{\text{st}})\). The toric moment polytope for \((\mathbb{CP}^2(1, 4, 5), \omega_{\text{st}})\) mutates to the toric moment polytope for \(\mathbb{CP}^2(a^2, b^2, 5c^2)\) if and only if \((a, b, c) \in \mathbb{Z}\) satisfies
\[
a + b^2 + 5c^2 = 5abc.
\]
In conclusion, a Symington Sequence for \((\text{Bl}_d(\mathbb{CP}^2), \omega_{\text{st}})\) yields any of the toric polytopes for \(\mathbb{CP}^2(1, 2, b, 5c^2)\), if \((b, c) \in \mathbb{Z}\) satisfy
\[
1 + b^2 + 5c^2 = 5bc.
\]
The sharp ellipsoid embeddings comes from the sequence of triples
\[
(1, 2, 1) \mapsto (1, 3, 1) \mapsto (1, 3, 2) \mapsto (1, 7, 2) \mapsto (1, 7, 5) \mapsto \ldots,
\]
which yields the sequence of \(E(1, 5/4), E(1, 9/5), E(1, 20/9), E(1, 49/20), E(1, 125/49)\)\( \ldots\), into an optimal rescaling of \((\text{Bl}_d(\mathbb{CP}^2), \omega_{\text{st}})\). This is a new sharp ellipsoid staircase, with sequence \(\{l_n\}_{n \in \mathbb{N}}\) of ratios of solutions converging to \(1 + \varphi\), where \(\varphi \in \mathbb{R}\) is the Golden Ratio.

### 3.3 Proof of Theorem 1.3

Let us now prove Theorem 1.3, building on the Symington sequences discussed in Subsections 3.1 and 3.2. For that, let us first state the result that we need from the theory of symplectic tropical curves, as developed in Section 4. The following statement is the only tropical ingredient for our proof of Theorem 1.3, and its proof will be the content of Section 4.

**Theorem 3.12** Let \(X(\mathcal{P}, \mathcal{I})\) be a symplectic toric domain, with \((\mathcal{P}, \mathcal{I}) \in \mathcal{H}\), an associated Symington sequence \((\mathcal{P}_n, \mathcal{I}_n)\), and let
\[
i_n : E(a_n, b_n) \to X(\mathcal{P})
\]
be a sharp symplectic ellipsoid embedding. For any \(\varepsilon \in \mathbb{R}^+\), there exists a tropical symplectic curve \(\tilde{S}_n \subseteq \mathcal{P}_n\) such that

(i) The tropical symplectic curve \(\tilde{S}_n \subseteq \mathcal{P}_n\) represents any embedded symplectic surface \(D(\tilde{S}_n) \subseteq X(\mathcal{P}_n)\) in the same homology class as an embedded symplectic curve in \(D(S)\),

(ii) There exists a neighborhood \(\mathcal{O}_p(\tilde{S}_n) \subseteq P_n\) and a volume-filling symplectic embedding
\[
i_n : E(a_n, b_n) \to X(\mathcal{P}_n) \setminus D(\tilde{S}_n).
\]

The inclusion \(\tilde{S}_n \subseteq \mathcal{P}_n\) in Theorem 3.12 is to be understood as \(\tilde{S}_n \subseteq P_n\) for a realization of the cut content \(\mathcal{B}_n\) of \(\mathcal{P}_n\). The exact realization of the cut content \(\mathcal{B}_n\), as cuts in \(P_n\), is a choice and, in Theorem 3.12, this choice depends on the initial value of \(\varepsilon \in \mathbb{R}^+\). Let us now apply Theorem 3.12 and conclude Theorem 1.3 in each of the cases.

**Complex Projective Plane** \(\mathbb{CP}^2\). There is a unique almost-toric base \(\mathcal{P} = (P, \mathcal{B})\) in Figure 1, with \(\mathcal{B} = \emptyset\). The relative almost-toric base \((\mathcal{P}, \mathcal{I})\) has \(D(\mathcal{I}) = \mathbb{CP}^1\), the complex projective line, as its unique symplectic divisor. This relative almost-toric base leads to the symplectic toric domain of the standard symplectic 4-ball
\[
X(\mathcal{P}, \mathcal{I}) = (\mathbb{D}^4(1), \omega_{\text{st}}) \cong (\mathbb{CP}^2 \setminus \mathbb{CP}^1, \omega_{\text{st}}).
\]
Let \( c_n = \text{Vol}(E(1,F_{n+1}^2/F_n^2)) \), with \( \{F_n\}_{n \in \mathbb{N}} \) as in Subsection 3.2, and let us show that there exists a sharp symplectic embedding

\[
i_n : E(1,F_{n+1}^2/F_n^2) \longrightarrow (\mathbb{D}^4(1), c_n \cdot \omega_{st}).
\]

The sequence of polytope mutations, as discussed in Subsection 2.2, associated to the Symington sequence in Subsection 3.1 yields a sequence of sharp symplectic embeddings

\[
i_n : E(1,F_{n+1}^2/F_n^2) \longrightarrow (\mathbb{CP}^2, c_n \cdot \omega_{st}).
\]

This sequence follows the arithmetic in Subsection 3.2 stemming from the Markov Equation with \( a = 1 \). In order to guarantee an embedding into \( X(\mathcal{P}, \mathcal{S}) \), it suffices to show that \( D(\mathcal{S}) = \mathbb{CP}^1 \) can be symplectically isotoped to lie above an arbitrarily small \( \varepsilon \)-neighborhood of the side opposite to the frozen vertex \( v_f \in \mathcal{P}, n \in \mathbb{N} \). This is achieved by first constructing a symplectic \( \mathbb{CP}^1 \) above this \( \varepsilon \)-neighborhood, and then arguing that this symplectic \( \mathbb{CP}^1 \) can be symplectically isotoped to the standard \( D(\mathcal{S}) = \mathbb{CP}^1 \subseteq X(\mathcal{P}) \). The former part is achieved by Theorem 3.12, which is also depicted explicitly in Figure 3 for the first five mutations in the Fibonacci Symington Sequence.

![Figure 3: Symplectic-tropicalCurves for \((\mathbb{CP}^2, \omega_{st})\).](image)

The latter part, constructing a symplectic isotopy from the lift of the symplectic tropical curve to the standard complex line \( D(\mathcal{S}) \), is now achieved by using M. Gromov’s [27], which shows that the symplectic isotopy class of the complex line is unique.

**The Ruled Surface** \((\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{st} \oplus \omega_{st})\). There are two (relative) almost-toric bases \((\mathcal{P}, \mathcal{S})\) in Figure 1, associated to the symplectic domains \( \mathbb{D}^2(1) \times \mathbb{D}^2(1) \), the polydisk, and the ellipsoid \( E(1,2) \). The first almost-toric base \((\mathcal{P}, \mathcal{S})\), on the right, has underlying polytope \( P \in \mathcal{P} \) the toric moment polytope for the weighted projective plane \( \mathbb{CP}^2(1,1,2) \). The cut content \( \mathcal{B} \in \mathcal{P} \) is non-trivial at one of the vertices, where a Lagrangian 2-sphere \( L(\mathcal{B}) \) lies above the segment unifying the two nodal points. This Lagrangian 2-sphere is in the homology class of Lagrangian anti-diagonal in \( X(\mathcal{P}) \cong (\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{st} \oplus \omega_{st}) \), and it is in fact Hamiltonian isotopic to the anti-diagonal. The surface configuration \( D(\mathcal{S}) \) consists of two surfaces: a symplectic 2-sphere, in the homology class of the complex line \( \{pt\} \times \mathbb{CP}^1 \), and the Lagrangian 2-sphere \( L(\mathcal{B}) \). The sequence of polytope mutations constructed in Subsection 3.1 yields a sequence of embeddings

\[
i_n : E(1,p_n) \longrightarrow (\mathbb{CP}^2(1,1,2), \text{Vol}(E(1,p_n)) \cdot (\omega_{st} \oplus \omega_{st})),
\]
where $p_n = \max \{ 2p_n^2/H_n^2, H_{n+1}^2/(2p_n^2) \}$, according to the arithmetic discussed in Subsection 3.2.

In order for these embeddings to be into $X(\mathcal{P}) = E(1, 2)$, instead of $X(P) = \mathbb{C}P^2(1, 1, 2)$, we first apply Theorem 3.12 and construct a symplectic tropical curve inside an $\varepsilon$-neighborhood of the edge opposite to the frozen vertex, which includes the cuts in $\mathcal{B}$. This tropical curve $\tilde{S}_n$ is chosen such that $D(\tilde{S}_n)$ is a symplectic 2-sphere in the homology class $\{ pt \} \times \mathbb{C}P^1$. Since the symplectic isotopy class of $\{ pt \} \times \mathbb{C}P^1 \subseteq \mathbb{C}P^1 \times \mathbb{C}P^1$ is unique, there exists a symplectic isotopy from $D(\tilde{S}_n)$ to $\{ pt \} \times \mathbb{C}P^1$.

The image of the Lagrangian sphere $L(\mathcal{B}_n)$ under this symplectic isotopy might not coincide with $L(\mathcal{B})$. Nevertheless, since $L(\mathcal{B}_n)$ and $L(\mathcal{B})$ are both Lagrangian 2-spheres in $\mathbb{C}P^1 \times \mathbb{C}P^1$ in the class of the anti-diagonal, there exists a Lagrangian isotopy connecting them, relative to the divisor $\{ pt \} \times \mathbb{C}P^1$ [33]. Since an exact Lagrangian isotopy can be realized by a Hamiltonian isotopy, this yields a symplectic isotopy from our surface configuration $D(\tilde{S}_n)$ and $L(\mathcal{B}_n)$ to the standard $D(S)$ and $L(\mathcal{B})$. This constructs the sharp symplectic embeddings

$$i_n : E(1, p_n) \longrightarrow (E(1, 2), \text{Vol}(E(1, p_n)) \cdot (\omega_{\text{st}})),$$

The second almost-toric base, on the left, corresponds to the toric moment polytope for $\mathbb{C}P^1 \times \mathbb{C}P^1$ and has $\mathcal{B} = \emptyset$. This yields the polydisk $X(\mathcal{P}) = \mathbb{D}^2(1) \times \mathbb{D}^2(1)$. Indeed, the surface configuration $D(\mathcal{P})$ for this first $(\mathcal{P}, \mathcal{P})$, depicted in blue, consists of a chain of two symplectic embedded 2-spheres, corresponding to $H_1 = \mathbb{C}P^1 \times \{ pt \}$ and $H_2 = \{ pt \} \times \mathbb{C}P^1$ respectively. Thus the the divisor $D = H_1 + H_2$ is defined as $\langle (\mathbb{C}P^1 \times \{ pt \}) \cup (\{ pt \} \times \mathbb{C}P^1) \rangle \subseteq X$, which represents the element $(1, 1) \in H^2(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{Z}) \cong \mathbb{Z}^2$. At this stage, we can directly invoke [11], which states that a symplectic ellipsoid embeds sharply in $E(1, 2)$ if and only if it embeds sharply in $\mathbb{D}^2(1) \times \mathbb{D}^2(1)$. This concludes Theorem 1.3 for these two almost-toric bases.

**Explicit Argument for** $\mathbb{D}^2(1) \times \mathbb{D}^2(1)$. Instead of [11], we can conclude Theorem 1.3 for this second almost-toric base as follows. In line with our argument in the first case of $\mathbb{C}P^2$, the question is whether a symplectic configuration $D(S) \subseteq \mathbb{C}P^1 \times \mathbb{C}P^1$ of a plumbing of two symplectic 2-spheres, representing the same cohomology class $[D(S)] = [D]$, is symplectic isotopic to $D$, through symplectic configurations which are plumbing of two symplectic 2-spheres. This is the notion of equisingular isotopy in [25, Definition 2.6], and D. McDuff’s work [43, 51] implies the answer in the affirmative.

For completeness, let us directly argue the existence of an equisingular isotopy from $D(S)$ to $D = H_1 + H_2$ as follows. Blow-up the intersection point of the two 2-spheres in $D$, and blow-down each of the proper transforms of the two original 2-spheres, which are themselves 0-self-intersection 2-spheres in $B_1(\mathbb{C}P^1 \times \mathbb{C}P^1)$. In the identification $B_1(\mathbb{C}P^1 \times \mathbb{C}P^1) \cong B_2(\mathbb{C}P^2)$, the exceptional divisor $E \subseteq B_1(\mathbb{C}P^1 \times \mathbb{C}P^1)$ blows-down to a (+1)-intersection symplectic 2-sphere $d(E) \subseteq \mathbb{C}P^2$. The uniqueness of the symplectic isotopy class of the complex line in $\mathbb{C}P^2$ gives an isotopy from $d(E)$ to any standard complex line $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$. This yields a symplectic isotopy, after blowing up twice and down once, from the symplectic configuration $D(S) \subseteq \mathbb{C}P^1 \times \mathbb{C}P^2$ to the standard configuration $\langle (\mathbb{C}P^1 \times \{ pt \}) \cup (\{ pt \} \times \mathbb{C}P^1) \rangle \subseteq \mathbb{C}P^1 \times \mathbb{C}P^2$.

The reminder of the argument is identical to that of the previous case, with Theorem 3.12 properly applied to construct symplectic representatives of $H_1$ and $H_2$ in an $\varepsilon$-neighborhood of the edge opposite to the frozen vertex. As explained in Subsection 3.2, we obtain sharp symplectic embeddings

$$i_n : E(1, p_n) \longrightarrow (\mathbb{C}P^1 \times \mathbb{C}P^1, \text{Vol}(E(1, p_n)) \cdot (\omega_{\text{st}} \oplus \omega_{\text{st}})),$$
where \( p_n = \max \{ 2p_n^2/H_n^2, H_n^2/(2p_n^2) \} \). Theorem 3.12 yields tropical representatives arbitrarily near the polytope side opposite from the frozen vertex. The symplectic isotopy discussed above then constructs the required sharp symplectic embeddings

\[
\iota_n : E(1, p_n) \to (\mathbb{D}^2(1) \times \mathbb{D}^2(1), \text{Vol}(E(1, p_n)) \cdot (\omega_{\text{st}} \oplus \omega_{\text{st}})).
\]

**Thrice Blown-up** \( Bl_3(\mathbb{CP}^2) \). There are four relative almost-toric basis \( (\mathcal{P}, \mathcal{S}) \) with \( X(\mathcal{P}) \cong Bl_3(\mathbb{CP}^2) \) symplectomorphic to the blow-up of \( (\mathbb{CP}^2, \omega_{\text{st}}) \) at three non-collinear points. In line with the previous case of symplectic domains \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), there exists an infinite sharp sequence of ellipsoids for one of these four symplectic domains \( X(\mathcal{P}, \mathcal{S}) \) if and only if it exists for one of them \([9]\). The numerics of these staircases will also coincide. The rightmost relative almost-toric base yields the symplectic ellipsoid \( E(2, 3) \cong X(\mathcal{P}, \mathcal{S}) \), with the unique symplectic sphere in \( D(\mathcal{S}) \) in the homology class of an exceptional divisor \( E \subseteq Bl_3(\mathbb{CP}^2) \). Let us focus on this almost-toric base. Consider the Symington sequence \( \mathcal{P}_n \), constructed in Subsection 3.2, associated to \( \mathbb{CP}^2(1, 2, 3) \), which yields a sequence of sharp ellipsoid embeddings

\[
i_n : E(1, k_n) \to Bl_3(\mathbb{CP}^2), \text{Vol}(E(1, k_n)) \cdot \omega_{\text{st}}).
\]

Theorem 3.12 now constructs an almost-tropical curve \( \tilde{S}_n \subseteq P_n \) such that \( D(\tilde{S}_n) \) is an embedded symplectic 2-sphere in the homology class \([E]\). At this stage, then we proceed as before, by constructing a symplectic isotopy from \( D(\tilde{S}_n) \) to the original exceptional divisor \( E \subseteq Bl_3(\mathbb{CP}^2) \), which is itself given by \( D(S) \), where \( S \) is the side of the toric moment polytope of \( \mathbb{CP}^2(1, 2, 3) \) opposite to the frozen smooth vertex. The symplectic isotopy embeds the complement of \( D(\tilde{S}_n) \) into the complement of \( E \), by upgrading it to an ambient symplectic isotopy, which is possible since the symplectic 2-sphere are embedded. As in the case of \( E(1, 2) \), the symplectic ellipsoid \( E(2, 3) = X(\mathcal{P}, \mathcal{S}) \) arises as the complement of \( E \) and three Lagrangian 2-spheres; this is the geometric incarnation of \( \mathcal{S} \) containing surfaces above the cuts. Similar to the case of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), the Hamiltonian isotopy classes of these Lagrangian 2-spheres in \( Bl_3(\mathbb{CP}^2) \) are known to be unique \([6, 47]\) and the Lagrangian isotopy can be taken to be in the complement of a stable symplectic sphere configuration \([46]\). The composition of our symplectic isotopy with a compactly supported Hamiltonian isotopy, bringing these three Lagrangian 2-spheres to our standard configuration in \( X(\mathcal{P}, \mathcal{S}) \), yields a sharp symplectic embedding

\[
\iota_n : E(1, k_n) \to (E(2, 3), \text{Vol}(E(1, k_n)) \cdot \omega_{\text{st}}).
\]

**Remark 3.13** The argument above requires an understanding of the Lagrangian spheres in \( Bl_3(\mathbb{CP}^2) \), which fortunately exists. Nevertheless, it is possible to instead argue with the leftmost relative almost-toric base \( (\mathcal{P}, \mathcal{S}) \) in Figure 1 directly as follows. In this case, \( \mathcal{S} \) is a unique symplectic configuration realized by a linear plumbing of four symplectic 2-spheres each with self-intersection \((-1)\), and representing the classes \( E_1, H, E_2 \) and \( H \), in this order. Theorem 3.12 applied to this singular configuration, yielding a tropical curve in an \( \epsilon \)-neighborhood of the side opposite of a frozen vertex. Then we need to prove that any such symplectic configuration \( S_1, S_2, S_3, S_4 \subseteq Bl_3(\mathbb{CP}^2) \) is equisingularly symplectic isotopic to a given such configuration \( D \) in the same homology class. For that, blow-down the second and fourth spheres \( S_2, S_4 \subseteq Bl_3(\mathbb{CP}^2) \). The blow-down of \( S_1 \) becomes a 0-self-intersection 2-sphere, and the blow-down of \( S_3 \) is a \((+1)\)-self-intersection 2-sphere in \( Bl_1(\mathbb{CP}^2) \). The classification of ruled symplectic surfaces \([43, 51]\) gives a symplectomorphism of \( Bl_1(\mathbb{CP}^2) \) that sends the symplectic configuration \( S_1 \cup S_3 \) to the union of the proper transform of the complex line \( \mathbb{CP}^1 \subseteq \mathbb{CP}^2 \) and the 0-self-intersection fiber of the linear pencil of \( \mathbb{CP}^2 \). Since the symplectomorphism group of \( Bl_1(\mathbb{CP}^2) \) is connected \([46]\), this can be achieved via a symplectic isotopy. This same symplectic isotopy, upon
blowing up twice, yields the required symplectic isotopy. The argument for Theorem 3.12 for this singular configuration should be in line with Section 4.

**Four Times Blown-up** \( Bl_4(\mathbb{C}P^2) \). The argument follows the same pattern as the previous three cases. Consider generators \((H, E_1, E_2, E_3, E_4) \in H^2(Bl_4(\mathbb{C}P^2), \mathbb{Z})\) given by the proper transform of the complex line in \( \mathbb{C}P^2 \), away from the blow-up points, and \( E_i \) the exceptional divisors, \( 1 \leq i \leq 4 \).

Let us analyze its leftmost relative almost-toric base \((P, S)\) in Figure 1, with \( D(S) \) a surface configuration consisting of a symplectic linear chain \( C \) of three self-intersection-\((-1)\) spheres, in the ordered homology classes \([E_1], [H] - [E_1] - [E_4], [E_1], [E_4]\), and two Lagrangian spheres in the homology classes \([E_3] - [E_1]\) and \([E_4] - [E_2]\). In the same scheme as before, the Symington sequence \( P_n \) associated to \( Bl_4(\mathbb{C}P^2) \), with arithmetic as in Subsection 3.2, yields sharp symplectic embeddings \( i_n : E(1, l_n) \longrightarrow (Bl_4(\mathbb{C}P^2), \operatorname{Vol}(E(1, l_n)) \cdot \omega_{st}) \).

Theorem 3.12 yields an almost-toric tropical curve \( \tilde{S}_n \) for each of these almost-toric bases \( \mathcal{P}_n \), equisingular to the symplectic configuration \( C \subseteq Bl_4(\mathbb{C}P^2) \) and with each 2-sphere in the same homology class as the 2-spheres in \( C \). Now, the equisingular symplectic isotopy class of these configuration is unique, by the same type of argument as in the cases of \( Bl_4(\mathbb{C}P^2) \) in Remark 3.13, and in \( \mathbb{C}P^1 \times \mathbb{C}P^1 \).

Indeed, blowing-down the two extremal 2-spheres in the classes \([E_1]\) and \([E_4]\) gives a \((+1)\)-symplectic 2-sphere in \( Bl_2(\mathbb{C}P^2) \) disjoint from the remaining two exceptional divisors. This provides a symplectic isotopy in \( Bl_2(\mathbb{C}P^2) \) from our configuration \( D(\tilde{S}_n) \) to the standard configuration in Figure 1. Since Lagrangian 2-spheres in \( Bl_4(\mathbb{C}P^2) \) are unknotted, and unique in their homology classes (up to Hamiltonian isotopy), the two Lagrangian 2-spheres can also be Hamiltonian isotoped to the standard configuration. This yields the required sharp embeddings

\[ i_n : E(1, l_n) \longrightarrow (Bl_4(\mathbb{C}P^2), \operatorname{Vol}(E(1, l_n)) \cdot \omega_{st}) \].

This concludes the proof of Theorem 1.3 assuming Theorem 3.12. Section 4 shall now present the proof of Theorem 3.12.

### 4 Symplectic Tropical Curves in Almost Toric Fibrations

This section develops the construction of configuration of symplectic curves in terms of almost-toric tropical diagrams. Theorem 3.12 and Theorem 1.3 are established. In a nutshell, tropical curves in almost-toric diagrams consist of two pieces: tropical curves in toric diagrams, as developed by G. Mikhalkin [56, 57], and tropical local models near the cut singularities of the affine structure. These second class of new local models are discussed in this section. There are two useful perspectives, from the viewpoint of contact 3-manifolds or directly from the perspective of symplectic 4-manifolds.

#### 4.1 An Introduction to Almost-Toric Tropical Curves

To start, we present the simplest of these local models by studying transverse links in contact 3-manifolds. The rest of the section will emphasize the local models directly in the symplectic 4-manifold. The case of almost-toric tropical curves in mutations of the toric moment polytope for \((\mathbb{C}P^2, \omega_{st})\) already contains the fundamental ideas for connected almost-toric tropical diagrams, and thus we now focus on this example. The reader is referred to [14, 24] for the required background in contact topology and [15, 16, 60] for the 4-dimensional symplectic fillings that shall appear.
Figure 4: On the left, an almost-toric tropical curve in an almost-toric base for \((\mathbb{CP}^2, \omega_{st})\). The weights of the vertices are determined by the arithmetic of the Symington sequence in Subsection 3.2. The region in dark blue is G. Mikhalkin’s model for a holomorphic tropical curve, whereas the region in light blue is the local model we need to discuss. On the right, the Legendrian link whose positive transverse push-off represents the (lift of the) intersection of the tropical curve with the contact 3-dimensional lens space, which is the boundary of a neighborhood of a cut.

Let us explain how to construct an embedded symplectic curve from the almost-toric tropical diagrams \(\tilde{S}\) in Figure 2 and the left of Figure 4. The diagrams for \(\tilde{S}\) are obtained by gluing to parts: a (strictly) tropical diagram and specific diagrams near the nodes and cuts. Since Theorem 3.12 is always applied in an \(\varepsilon\)-neighborhood of the side opposite to the frozen vertex, it suffices to consider a neighborhood of this side. The left of Figure 4 depicts such a neighborhood. In order to represent the symplectic class of the projective line \(\mathbb{CP}^1 \subseteq \mathbb{CP}^2\), the weight of the tropical trivalent vertex for the leg on that side must be three, since \([K_{\mathbb{CP}^2}] = 3 \cdot [\mathbb{CP}^1]\). The remaining two weights are determined by the combinatorics of the polytope, and are readily shown to be \((F_n, F_{n+1})\), following the numerics in Subsection 3.2.

Figure 5: Each column depicts an example of a Legendrian handlebody for a rational homology ball \(B_{n,m}\), with its contact boundary lens space \(L(n^2, nm - 1)\). The Legendrian handlebody is a contact surgery diagram for the contact 3-manifold above the red curve in the almost-toric diagram. The left column depicts \(L(4, 1)\) as a boundary of \(B_{2,1} \cong T^*\mathbb{RP}^2\), and the right column depicts \(L(25, 9)\) as a boundary of \(B_{5,2}\). The fractions in the almost-toric diagrams are the slopes of the segments.
The question is now completing the piece of (actual) tropical curve away from the cuts to an almost-toric tropical curve. In terms of Figure 4 (Left), we first construct the dark blue region away from the cuts, and we aim at building an embedded symplectic curve above the light blue pieces. The contact 3-dimensional approach is the following.

Consider the intersection of the tropical piece with the 3-dimensional contact lens space 
\[(L(k \cdot n^2, k \cdot nm - 1), \xi_{st})\]
which lies above the red curve on the diagram of the left of Figure 4. The number \(k \in \mathbb{N}\) indicates the number of nodes, and thus \(k = 1\) for any vertex appearing in a mutation of \((\mathbb{C}P^2, \omega_{st})\).

**Remark 4.1** It is immediate that the closed 3-manifold is a lens space, as it admits a genus one Heegaard splitting, with the Heegaard surface given by a torus fiber of the almost-toric fibration. The precise \(n, m, k \in \mathbb{N}\) are determined by the slopes of the sides. In the case of one node \(k = 1\), if one side has infinite slope, then the cut will have slope \(m/n\) and the second side shall have slope \((nm - 1)/n^2\).

Figure 5 depicts two instances of the lens spaces that appear in the first two mutations for \((\mathbb{C}P^2, \omega_{st})\). For \(L(1, 1) \cong (S^3, \xi_{st})\) the arriving weights will be 1 and 2, and for \(L(4, 1)\) the arriving weight shall be 5, confer Figure 2. The intersection of the tropical curve with this 3-dimensional contact manifold is generically a transverse link, with as many components as the weight of the leg. From the contact viewpoint, the construction of the light blue piece is now tantamount to finding a symplectic surface filling of a transverse link in \((L(k \cdot n^2, k \cdot nm - 1), \xi_{st})\) inside the 4-dimensional symplectic filling given by the pre-image of the region in the almost-toric diagram which contains the cut and is bounded by the red curve. This transverse link can be readily described as a positive transverse push-off of a Legendrian link. In particular, an exact Lagrangian filling for such a Legendrian link shall yield an embedded symplectic surface for its transverse push-off.

In general, a Legendrian link does not admit exact Lagrangian fillings and, even if it does, their topology is constrained. Fortunately, the Legendrian links appearing in this context are obtained by taking the \(w\)-copy Reeb push-off of the attaching boundary of the defining 2-handle for the 4-dimensional symplectic filling, where \(w \in \mathbb{N}, w \geq 2\), is the arriving weight. For instance, a leg with weight 5 arriving to \(L(4, 1)\) with the slope as in Figure 2 intersects \((L(4, 1), \xi_{st})\) at the positive transverse push-off of the 5-component Legendrian link in Figure 4 (Right). Since the Legendrian link consists of a \(w\)-copy of a Reeb push-off, it will always bound a \(w\)-punctured exact Lagrangian sphere in the symplectic filling given by these Legendrian handlebody diagrams. Indeed, performing \((w - 1)\)-pinching moves, connecting the different components to one, yields the standard Legendrian unknot, which locally bounds a flat Lagrangian disk. The \(w\)-punctured Lagrangian sphere is precisely obtained as this Lagrangian disk with \((w - 1)\) Lagrangian 1-handles attached.

**Example 4.2** The simplest example of the above construction is building an almost-toric tropical curve for the standard toric diagram for \((\mathbb{C}P^2, \omega_{st})\) with two cuts introduced in two of the toric vertices. Figure 6 depicts this situation.

The transverse knot in this situation is the standard transverse unknot, with maximal self-linking number. It is depicted in orange in the Legendrian handlebody for the symplectic filling \((\mathbb{D}^4, \omega_{st})\) given by one 1-handle and one 2-handle.
Note that the topology of the resulting symplectic surface is that of a 2-sphere. Indeed, the tropical piece consists of a 2-sphere with \((w_1 + w_2)\)-punctures, given by the weights \(w_1, w_2\) of the two legs which do not intersect the anti-canonical toric divisor. The local surface models introduced above, obtained by deforming the Lagrangian fillings to a symplectic surface, are two symplectic 2-disks with \(w_1\) and \(w_2\) punctures. The gluing of these pieces yields a closed symplectic 2-sphere.

In general, the lens spaces shall be of the form \(L(k \cdot n^2, k \cdot nm - 1)\) and, for \(k \geq 2\), there might be a choice of Lagrangian filling. These different choices always allow us to adjust the homology class as desired. The reader is invited to work the case \(k = 2\) in detail, where the 4-dimensional symplectic filling contains a Lagrangian 2-sphere, along which one can Dehn twist. Let us now turn gears towards the 4-dimensional approach, where the local models shall be constructed directly.

### 4.2 Almost-toric Fibrations

In this subsection, we succinctly discuss almost-toric diagrams and their associated almost-toric fibrations, henceforth abbreviated ATF \([65]\), with the focus on visualizing symplectic surfaces, curves in complex dimension one, on a given ATF. We refer the reader to Section 2 and \([45, 65]\) for additional details. The central definition, \([65, \text{Definition 4.5}]\) and \([45, \text{Definition 2.2}]\), reads as follows:

**Definition 4.3** \(([45, 65])\) An almost-toric fibration of a symplectic 4-manifold \((X, \omega)\) is a Lagrangian fibration \(\pi : (X, \omega) \to B\) such that any point of \((X, \omega)\) there exists a Darboux neighborhood \((x_1, x_2, y_1, y_2) \in (\mathbb{D}^4, \omega_{\text{st}})\), with symplectic form \(\omega_{\text{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2\), in which the map \(\pi\) has one of the following local normal forms:

\[
\begin{align*}
\pi(x, y) &= (x_1, x_2), & \text{regular point}, \\
\pi(x, y) &= (x_1, x_2^2 + y_2^2), & \text{elliptic, corank one}, \\
\pi(x, y) &= (x_1^2 + x_2^2 + y_2^2), & \text{elliptic, corank two}, \\
\pi(x, y) &= (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1), & \text{nodal or focus-focus},
\end{align*}
\]

with respect to some choice of coordinates near the image point in the almost-toric base \(B\). \(\square\)

For the nodal singularity, writing \(x = x_1 + ix_2\) and \(y = y_1 + iy_2\), the almost-toric fibration \(\pi\) reads \(\pi(x, y = \bar{xy} \in \mathbb{C}\). The following remark is also relevant:
Remark 4.4 By the Arnold-Liouville Theorem [3], given any point \( q \) in a regular fibre, there is a neighborhood of the form \( U \times T^2 \), with action-angle coordinates \((p_1, p_2, \theta_1, \theta_2)\), \((p_1, p_2) \in \mathbb{R}^2\) and \((\theta_1, \theta_2) \in T^2 = \mathbb{R}^2 / \mathbb{Z}^2\), where the symplectic for \( \omega \) is \( dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2 \). Here, we identify simultaneously \( U \) with a neighborhood of \( \pi^{-1}(q) \) in \( B \), as well as, a neighborhood of \( \mathbb{R}^2 \). Consider then a 1-cycle \( \gamma(t) = (\theta_1(t), \theta_2(t)) \in \pi^{-1}(q) = \mathbb{R}^2 / \mathbb{Z}^2 \) and move it along a curve \( \sigma(s) \) in \( U \subset \mathbb{R}^2 \), so we get the cylinder \( \Gamma(s, t) = (\sigma(s), \gamma(t)) \) in the \((p_1, p_2, \theta_1, \theta_2)\) coordinates. So, up to a choice of orientation, \( \Gamma(s, t) \) is symplectic iff \( \langle \sigma'(s) | \gamma'(t) \rangle \neq 0 \) for all \( t \). In particular, if we fix the cycle with \( \gamma'(t) = (a, b) \), \((a, b) \in \mathbb{Z}^2\), we are allowed to move in any direction in \( U \), as long as we are never parallel to \( \langle (b, -a) \rangle \), in order to get a symplectic cylinder. \( \square \)

The set \( B_0 \) of regular values of \( \pi : X^4 \to B^2 \), carries naturally an affine structure, and circling around a node, i.e. the image of a nodal critical point, provides a monodromy for this affine structure, that is a shear with respect to some eigendirection associated to the node – see [45, 65] for details. In local action-angles coordinates (see Remark 4.4), where we have \( \pi|_V : V \cong U \times T^2 \to U \subset \mathbb{R}^2 \), with \( U \hookrightarrow B \), this affine structure is identified with the lattice \( \mathbb{Z}^2 \subset \mathbb{R}^2 \cong T_b U \) for each \( b \in U \).

As in Section 2, an ATF can be described by an almost-toric base diagram, which in section is denoted by \( \Delta \subset \mathbb{R}^2 \) (ATBD) [65, Section 5.2]. This is the reason for Definition 2.1. Indeed, in this paper an ATBD is depicted as a polytope, with nodes in the interior representing the nodal fibres, and cuts (represented by dashed segments) towards the edge, that encode the monodromy around the singular fibres.

Remark 4.5 Figure 7 represents ATBD representing an ATF of \((\mathbb{C}P^2, \omega_{st})\), see also Figure 2 and Figure 1, and [69] for several ATBDs representing ATFs of Del Pezzo surfaces. In this article, we will always assume that the any cut in the ATBD is a segment in the eigendirection of the monodromy of its associated node. That way we can define a homeomorphism \( B \to \Delta \), which is affine over \( B_0 \setminus \bigcup \text{(cuts)} \). To each cut, we associate a monodromy matrix in \( \mathcal{L}(2, \mathbb{Z}) \), representing, in the standard basis of \( \mathbb{R}^2 \), the monodromy associated to a counter-clockwise loop around the node. \( \square \)

The symplectic curves we construct project to a neighborhood of a graph embedded in the base of the ATF, in line with Mikhalkin’s amoebas projecting to the neighborhoods of a tropical spine [55, 54, 59]. In our case, we have more flexibility since we only require the curve to be symplectic, not necessarily holomorphic. A series of recent works by several authors [48, 49, 58, 31] shows how to get Lagrangians from tropical data, which is also much more rigid then getting symplectic curves. In order to describe symplectic tropical curves, we develop terminology in Subsection 4.3.

Figure 7: ATBD for \( \mathbb{C}P^2 \).
4.3 Symplectic-tropical curves

The central ingredient in Theorem 3.12 and in our argument in Subsection 3.3 is the notion of a *symplectic tropical curve* in an almost-toric diagram, which we abbreviate STC. This is the content of the following definition.

Let $\Delta \subset \mathbb{R}^2$ be an ATBD representing an ATF $\pi : X \to B$ of a symplectic 4-manifold $X$. Let $\Gamma$ be an oriented graph, with edges decorated by primitive $\mathbb{Z}^2$ vectors and a multiplicity in $\mathbb{Z}_{>0}$. Given an oriented edge $\gamma$ from a vertex $b$ to a vertex $c$, $b$ is said to be negative and $c$ positive, with respect to $\gamma$.

**Definition 4.6** A symplectic tropical curve, or a symplectic-tropical curve, $\mathcal{C} : \Gamma \to \Delta$ is a $C^0$-embedding which satisfies the following conditions:

(i) Vertices are either univalent (boundary), bivalent (bending) or trivalent (interior), [edges associated with boundary vertices we call leaves];

(ii) All boundary vertices are negative.

(iii) Images under $\mathcal{C}$ of boundary vertices are either on the boundary of $\Delta$ or on a node, of bending vertices are on the cuts (hence endow an associated monodromy matrix), of interior vertices are in the complement of cuts, nodes, and boundary;

(iv) $\mathcal{C}$ restricted to the (interior of) the edges is a $C^\infty$ embedding and tangent lines have lateral limits at each vertex, which are oriented according to the orientation of the corresponding edge [we call a vector on this limit tangent lines a limit vector];

(v) If $v$ is a positively oriented vector tangent to the image of an edge under $\mathcal{C}$, with associated primitive vector $w \in \mathbb{Z}^2$, then $\langle v | w \rangle > 0$;

(vi) For a boundary vertex over the boundary of $\Delta$, the primitive $\mathbb{Z}^2$ vector associated to its corresponding leaf must be orthogonal to the boundary and pointing towards the interior of $\Delta$ [the multiplicity can be arbitrary];

(vii) For a boundary vertex over a node of $\Delta$, the primitive $\mathbb{Z}^2$ vector $w$ associated to its corresponding leaf must be orthogonal to the cut, and its orientation is determined by (ii) and (v) [the multiplicity can be arbitrary];

(viii) Let $\gamma_1$ and $\gamma_2$ be two edges meeting at a bending vertex, with monodromy matrix $M$. First, the bending vertex must be positive w.r.t. one edge and negative w.r.t. the other. Assume that we go counter-clockwise from $\mathcal{C}(\gamma_1)$ to $\mathcal{C}(\gamma_2)$. If $v$ is a positively oriented limit vector for $\mathcal{C}(\gamma_1)$ at this bending vertex, then $Mv$ is a positively oriented limit vector for $\mathcal{C}(\gamma_2)$. Moreover, if $w$ is the associated primitive vector to $\gamma_1$, then the associated primitive vector to $\gamma_2$ must be $(M^T)^{-1}w$. Also the multiplicity of $\gamma_1$ and $\gamma_2$ must be the same;

(ix) For the three edges $\gamma_j$, $j = 1, 2, 3$, associated to an interior vertex $b$, with associated vectors $w_j$ and multiplicity $m_j$, the following balancing condition must be satisfied:

\[
\varepsilon_1 m_1 w_1 + \varepsilon_2 m_2 w_2 + \varepsilon_3 m_3 w_3 = 0,
\]

where $\varepsilon_j = \pm 1$ according to $b$ being positive or negative with respect to $\gamma_j$, $j = 1, 2, 3$. $\square$

There are many conditions in Definition 4.6, the idea is the following. We travel along an edge with cycles, according to the multiplicity, represented by its associated vector, as in Remark 4.4. Condition
(v) will guarantee that we are symplectic as we travel along the edges; condition (ix) implies that these cycles arriving at a vertex are null-homologous, which will allow us to glue them; conditions (vi) and (vii) guarantee that the corresponding cycles collapse as we arrive at an edge, or can collapse at the nodal fiber; condition (viii) says that the map $\Gamma \to B$ given by composing with the inverse of the homeomorphism $B \to \Delta$, is actually smooth at the bending vertex, and the associated vector changes accordingly. The remaining notation exists in order to write the balancing condition (4–1) in a precise and consistent way.

We represent a symplectic-tropical curve by just drawing its image $C(\Gamma)$, and labeling the multiplicity of each edge, that being 1 if unlabeled. Conditions (vi)–(ix), determine the associated vectors and multiplicity, up to the ambiguity given by the sign and orientation of the edges that are not leaves, which we basically ignore since they are just an artefact to write (ix) consistently.

For instance, Figure 8 represents 3 different symplectic lines in an ATF of $\mathbb{C}P^2$. In the leftmost picture, the leaves are decorated with the associated vectors $(1, -1)$, $(0, 1)$ and $(-1, -2)$, accordingly, while the other two edges have labels $\pm(1, 0)$ and $\pm(1, 1)$, the sign being determined by (ix) and how one decides to orient the edges.

**Proposition 4.7** Given $\mathcal{C} : \Gamma \to \Delta$ symplectic-tropical curve as above, and a neighborhood $\mathcal{N}$ of $\mathcal{C}(\Gamma)$, there exists a closed symplectic curve (2 real dimensions) $C$ embedded in $X$, projecting to $\mathcal{N}$ under $\pi : X \to B \to \Delta$. In addition, the intersection of $C$ with the anti-canonical divisor living over the boundary of $\Delta$ is given by the sum of the multiplicities of the corresponding boundary vertices. □

In this case, we say that $C$ is represented by $\mathcal{C}(\Gamma)$. See [65, Proposition 8.2] for a discussion on the anti-canonical divisor. In order to prove Proposition 4.7, we need to construct local models near the nodes and the trivalent vertices. This is the content of the following subsections.

**Remark 4.8** We could consider embedding of graphs with interior vertices having valency greater than 3, and write a rather involved definition for symplectic-tropical curves. In this case, these graphs could be viewed as a limit of graphs as in Definition 4.6, and a result similar to Proposition 4.7 holds. □

### 4.3.1 Local model near the nodes

The interesting cases near the nodes is that we may arrive at a node with multiplicity $k$ greater than 1. It is possible to address this using the transverse $k$-component links, as explained in Subsection 4.1.
We shall now argument directly in the 4-dimensional symplectic domain and monitor the projections onto the ATBD.

In principle, we would need to get \( k \) disjoint capping disks with boundary in \( k \) copies of the collapsing cycle nearby a nodal singularity. That is not possible if we force the boundary of these disks to live in some torus fibre. So, in the case \( k > 1 \), our disks then, will not project over a segment under \( \pi \), but rather to a \( 2d \) thickening of a segment.

Let us start looking at the easier case \( k = 1 \), where we are to collapse only one symplectic disk at the singular point. Consider the local model of a nodal fibre as in Definition 4.3, and the complex notation \( \pi(x, y) = \bar{xy} \) for the almost-toric fibration the map \( \pi \).

In this case, the disk \( \sigma_1 = (re^{i\theta}, -ire^{i\theta}), \ 0 \leq |r| \leq \varepsilon \ll 1, \ \theta \in [0, 2\pi] \) is a symplectic disk with boundary \( c(\theta) = (\varepsilon e^{i\theta}, -i\varepsilon e^{i\theta}) \), which is part of the symplectic line \( y = -ix \) and project via \( \pi = \bar{xy} \) to \( \bar{irR}_{\leq 0} \).

Now let us get a second symplectic disk in this neighborhood, disjoint from the above, so that its boundary is isotopic and very close to the collapsing cycle \( c(\theta) \). Fix a value \( \delta_2 \), thought to be very small with respect to \( \varepsilon^2 \), and a monotone non-increasing \( C^\infty \) bump function \( \Psi_2(s) \), that is \( \delta_2 \) for \( s \) close to zero and \( 0 \) for \( s \geq 1 \). Moreover, take \( \Psi_2(s) > 0 \) for \( s \in [0, \varepsilon^2] \). By taking \( \delta_2 \) sufficiently small, we can take \( \Psi \) as \( C^1 \) close to \( 0 \) as necessary. We can then guarantee that the line \( l_2 : \{ y = -ix + \Psi_2(|x|^2) \} \) is still symplectic. Our second disk \( \sigma_2 \), is the intersection of \( l_2 \) with the imaginary part \( \text{im}(\bar{xy}) \geq -\varepsilon^2 \). Note that \( \sigma_2 \cap \sigma_1 = \emptyset \).

In order to then construct \( k \) mutually disjoint symplectic disks, with their boundary boundary isotopic and very close to the collapsing cycle \( c(\theta) \), we only need to take \( \delta_k > \delta_{k-1} > \cdots > \delta_2 \), and the corresponding monotone non-increasing \( C^\infty \) bump functions \( \Psi_k(s) > \Psi_{k-1}(s) > \cdots > \Psi_2(s) \), all sufficiently \( C^1 \) close to \( 0 \) so that the lines \( l_j : \{ y = -ix + \Psi_j(|x|^2) \} \) are symplectic for \( j = 2, \ldots, k \). The disk \( \sigma_j \) is then the intersection of \( l_j \) with \( \text{im}(\bar{xy}) \geq -\varepsilon^2 \). See the leftmost picture of Figure 9 for a depiction of \( \pi(\sigma_j) \)'s.

**Remark 4.9** The equations for \( l_j \) imply that \( \sigma_j \) does intersect the Lagrangian \( x = y \), once for all \( j = 1, \ldots, k \), at a point where \( x = -ix + \Psi_j(|x|^2) \), since by the Phytagorean Theorem, we need \( \Psi_j(|x|^2) = \sqrt{2}|x| \), which occurs since we took \( \delta_j << \varepsilon^2 \).

Note that \( \partial \sigma_j \), projects under \( \pi \) to a segment \( I_j \) normal to \( \bar{irR}_{\leq 0} \). Using the fact that the \( l_j \) lines are isotopic to the line \( y = -ix \), which topologically self-intersects once (say relative to the boundary at infinity), we can deduce that \( \partial \sigma_j \) links \( \partial \sigma_1 \) once in the thickened annulus \( \pi^{-1}(I_k) \), as illustrated in the rightmost picture of Figure 9.

In proving Proposition 4.7, we may assume that the image of the leaves arrive at a node in a segment, which we can identify in our local model with a segment in \( \bar{irR}_{\leq 0} \). The above discussion is then summarized in the following

**Proposition 4.10** Let \( \pi : X \to B \cong \Delta \) be an ATF, represented by an ATBD \( \Delta \), where we homeomorphically identified \( B \) with \( \Delta \). Let \( \gamma \) be a leaf of a symplectic-tropical curve \( \mathcal{C} : \Gamma \to \Delta \), with boundary vertex over a node, and \( k \in \mathbb{Z}_{>0} \) is its given multiplicity. Fix a point \( p \in \mathcal{C}(\gamma) \) close to the node, and the collapsing cycle \( \alpha \subset L_p := \pi^{-1}(p) \). Let \( \mathcal{N} \) be a neighborhood of \( \mathcal{C}(\gamma) \). Then we can associate \( k \) symplectic disks \( \sigma_j, j = 1, \ldots, k \), so that, \( \pi'(\sigma_j) \subset \mathcal{N} \), with their boundary \( \partial \sigma_j \) arbitrarily close to \( \partial \sigma_1 = \alpha \).
4.3.2 Local modal near interior vertices

Condition (v) in Definition 4.6, and the comments on Remark 4.4, allow us to transport the boundary of the $\sigma_1$ disk of Proposition 4.10 along the image of the corresponding leaf until very close to a interior node (if we hit a bending vertex, condition (viii) guarantees that we can just keep moving the same cycle $\partial\sigma_1$, whose class in the first homology of the fibre is then represented by a different $\mathbb{Z}^2$ vector according to the monodromy). Also, $\partial\sigma_j$ projects to a segment normal to the leaf and because of its closeness of to $\partial\sigma_1$, we can also transport it using cycles projecting under $\pi$ to small segments normal to the leaf. Arguments similar to Remark 4.4 ensure that we remain symplectic.

Now we need to construct a local model for a surface near a trivalent vertex $b$, using the data of (ix), that can be made to project to a given neighborhood of $C(b)$. Moreover, it needs to glue with prescribed incoming cycles.

One option is to rely on [55], where G. Mikhalkin uses Viro’s patchworking ideas to construct families of hypersurfaces in $(\mathbb{C}^*)^n$, whose amoebae converge to a given tropical hypersurface in $\mathbb{R}^n$, see [55, Remark 5.2]. In the reference, the author actually views $(\mathbb{C}^*)^n$ as the open strata of a closed toric variety, in particular having finite volume, so we can symplectically assume that it is indeed a local model. Given the nature of Proposition 4.7, we can assume that the symplectic-tropical curve arrives at the interior vertex in a tropical way, i.e., locally as segments $v_j = w_j$’s satisfying the balancing condition (4–1) of Definition 4.6.

Nevertheless, we can also use the following explicit local model. Over a small disk $B$ centred at an interior vertex $b \in B$, we have $\pi^{-1}(B) \cong B \times T^2$ and the fibration is given by the projection. The projection of the symplectic curve to the $T^2$ factor is known as the coamoeba of that curve. We intend to first construct what will be the coamoeba out of the balancing condition (4–1) and out of that data build our local model for the symplectic curve. We want the surface to be so that its boundary projects to straight cycles, having only double crossings. Moreover, away from the pre-image of the double crossings, the rest of the surface will project injectively into polygons divided by the straight cycles. The homology classes of the boundary are represented by $m_j$ disjoint copies of $\varepsilon_j w_j$, $j = 1, 2, 3$, where $w_j$’s are the vectors associated with the interior vertex $b$ and hence satisfying the balancing condition (4–1) from Definition 4.6.

The leftmost picture in Figure 10 illustrates the amoeba of the local model we will build for the neighborhood of the interior vertex in the middle ATBD in Figure 8. The balancing condition associated with the interior vertex is $(1, -1) + 3(0, 1) + (-1, -2) = 0$.

The existence of the above mentioned configuration for the coamoeba is equivalent to the existence of a dimer model embedded in $T^2$ — a dimer model [40] is a bipartite graph, and we name half of the vertices

Figure 9: The left picture illustrates the projection of the $\sigma_j$ disks. The right picture illustrates how $\partial\sigma_j$ and $\partial\sigma_1$ are linked in the thickened annulus $\pi^{-1}(I_k)$. 

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black and the other half white, so edges connect a black vertex to a white one. We label each convex polygon of our coamoeba black and white, where a black polygon only share a vertex with a white one. The vertices of the dimer model are then placed in the interior of the polygons according to their colours, and for each intersection of the boundary cycles we associate an edge, projecting inside the coamoeba, see the rightmost picture of Figure 10. The straight cycles becomes than what is known as the collection of zigzag paths for that dimer. For concepts related to dimer models and its relationship with coamoebas, we refer the reader to the more recent works [17, 29, 21, 31].

Our aim is then to provide a dimer model with prescribed set of homology classes for its collection of zigzag paths. Moreover, we want the zigzag paths to be straight. Without this last condition, in [29, Section 6] Gulotta constructs an algorithm to build a dimer model out of the prescribed classes for the collection of zigzag paths, with non-straight cycles. In [21, Example 4.1], it is shown a counter-example that a given collection of 5 classes in $H_1(T^2)$ that cannot be realised by straight cycles that are the zigzag paths of a dimer model. However, if your collection of classes are given by copies of only 3 primitive classes, one can construct a dimer model with straight set of zigzag paths, as shown in Proposition 4.11 below. This is most likely known to experts but we did not find references in the literature.

**Proposition 4.11** Given $w_1, w_2, w_3 \in H_1(T^2; \mathbb{Z})$, primitive classes satisfying $m_1w_1 + m_2w_2 + m_3w_3 = 0$, there exists a dimer model in $T^2$ with zigzag paths realised by $m_j$ straight lines in the classes $w_j$, $j = 1, 2, 3$. Moreover, the component containing the vertices of the dimer can be taken to be triangles.

**Proof** Figure 11 is essentially a proof by drawing. Spelling the proof out, without loss of generality, we may assume $w_1 = (1, 0)$. Name $w_2 = (a_2, b_2)$, $w_3 = (a_3, b_3)$. Take the straight cycle $[0, 1] \times \{0\}$ in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, in the class $w_1 = (1, 0)$, and consider $|m_2b_2| = |m_3b_3|$ equidistant points in $[0, 1]$. For $j = 2, 3$, take $m_j$ straight lines with slope $w_j$, passing through the first $m_j$ points in the segment $[0, 1]$, and consider them as cycles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. It can be seen that these lines intersect in heights multiples of $1/m_1$ mod $\mathbb{Z}$. For $k = 0, \ldots, m_1 - 1$, take the cycles $[0, 1] \times \{k/m_1\}$, each of them intersecting the other straight lines in $|m_2b_2| = |m_3b_3|$ triple points. In that way, $T^2$ is divided into triangles, as illustrated by the left picture in Figure 11. Moving the horizontal cycles slightly up, we can build a dimer as illustrated in Figure 11.
Out of this dimer model constructed from the data of the balancing condition \((4-1)\), we will first build a smooth surface in \(I \times T^2\), where \(I = [-\epsilon, \epsilon]\) with boundary \(m_j\) copies of cycles in class \(w_j = \epsilon_j w_j\) in \(H_1(I \times T^2; \mathbb{Z}) \cong H_1(T^2; \mathbb{Z})\), living in heights \(-\epsilon, 0, \epsilon\) for \(j = 2, 1, 3\), respectively.

Consider the congruent white triangles and, in each, take a segment from its vertex given by the intersection of the cycles in the classes \(w_2\) and \(w_3\), to the middle point of the opposite edge. We name the primitive direction of this segment \(f = (\alpha, \beta)\) and often simplify notation by calling \(f\) the segment itself. Consider then a smooth foliation of the triangle minus the vertices, so that following it gives an isotopy from the edge in the \(w_2\) cycle to half of the edge in the \(w_1\) cycle union the segment \(f\), and then from the other half of the edge in the \(w_1\) cycle union the segment \(f\) to the the edge in the \(w_3\) cycle, as illustrated in the leftmost picture of Figure 12. Considering each leaf of the foliation as level sets of a smooth function \(\rho_2\) from the triangle triangle minus vertices to \(I\), its graph embeds into \(I \times T^2\), as illustrated by the rightmost picture of Figure 12. Taking a symmetric version of the foliation and function \(\rho_2\) on the black triangles of the dimer model, ensures that the compactification of the union of the graphs is a smooth surface in \(I \times T^2\), with the desired boundaries.
This embedding can be made symplectic into $B \times T^2$, the symplectic neighborhood of the pre-image of interior vertex $b$, by simply embedding $I$ into $B$ in an appropriate direction. Nonetheless, we can also build our surface to project on the $(p_1, p_2)$ coordinates onto amoebas, much like in the work of Mikhalkin [55, 54].

![Diagram of amoeba and coamoeba](image)

**Figure 13**: The depiction a triangular piece of the coamoeba (left) and the amoeba (right) of a symplectic surface near the interior vertex.

To prepare ourselves for the construction of symplectic surface we will change the local coordinates from $(p_1, \theta_1, p_2, \theta_2)$ to new coordinates $(\rho_1, \phi_1, \rho_2, \phi_2)$, keeping the symplectic form $dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2 = d\rho_1 \wedge d\phi_1 + d\rho_2 \wedge d\phi_2$.

We want to change the basis in the $T^2$ chart from $e_1 = (1, 0)$, and $e_2 = (0, 1)$, to $e_1$ and $f = (\alpha, \beta)$. This corresponds to setting $\phi_1 = \theta_1 - (\alpha/\beta) \cdot \theta_2$, and $\phi_2 = \theta_2 / \beta$, so $\theta_1 e_1 + \theta_2 e_2 = \phi_1 e_1 + \phi_2 f$. Setting $\rho_1 = p_1$ and $\rho_2 = \alpha p_1 + \beta p_2$ ensure that $dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2 = d\rho_1 \wedge d\phi_1 + d\rho_2 \wedge d\phi_2$.

We are now ready to prove:

**Proposition 4.12** Let $\mathscr{C} : \Gamma \to \Delta$ be a symplectic-tropical curve in an ATF $\pi : X \to B \cong \Delta$, represented by an ATBD $\Delta$, $b$ be an interior vertex and $B$ a small disk centered in $b$, whose boundary intersect $\mathscr{C}(\Gamma)$ in the points $p_1, p_2, p_3$. Let $w_j$ be the associated vector corresponding to the edge containing $p_j$, for $j = 1, 2, 3$, hence satisfying the balancing condition (4–1). There is a symplectic curve in $X$, projecting into $B$, whose boundary project to the points $p_j$’s and are cycles, whose classes in $H_1(p_j)$ are represented by $w_j$.

**Proof** Figure 13 illustrates how the embedding will look like in each of the (white) triangles, by describing the amoeba and coamoeba together. The interior vertex corresponds to $(\rho_1, \rho_2) = (0, 0)$, and we vary $\rho_1$ in a very small interval $[-\delta, \delta]$. The smooth surface we construct will be the projection to the $(\rho_2, \phi_1, \phi_2)$ coordinates. Hence, our foliation in the triangle corresponds to constant $\rho_2$. Each edge of the triangle in the coamoeba have constant $(\rho_1, \rho_2)$ coordinates corresponding to the vertices of the amoeba. Using the same name for the edges, as for their homology classes, we choose $\rho_1$, so that, $\rho_1(w_3) = \rho_1(w_2) < 0 \leq \rho_1(w_1)$. The $\rho_1$ coordinate decreases as we move along the $\rho_2$ level sets from bottom to top. Also, as we vary $\phi_1$ positively in the horizontal segments ($\phi_2$ constant) in the triangle, we choose $\rho_1$ so the its variation is non-negative for $\rho_2 \leq 0$ and non-positive for $\rho_2 \geq 0$; also the $\rho_2$ coordinates vary positively. We also note that the segments in our smooth surface that projected to the vertices of the triangle in $T^2$, will project to the boundary of the amoeba in $B$. 


To check the symplectic condition, we probe with the paths $\xi^1$, $\xi^2$, the former given by following the horizontal segment ($\phi_2$ constant) in the triangle, and the later given by following the $\rho_2$ level sets in the triangle. Denoting by $d\rho_i^j$ and $d\phi_i^j$ the coordinates of $d\rho_i$ and $d\phi_i$, along the $j$-th path, for $i,j=1,2$, we have that,

$$d\rho_1^1 = \begin{cases} \geq 0 & \text{if } \rho_2 \leq 0 \\ \leq 0 & \text{if } \rho_2 \geq 0 \end{cases}, \quad d\phi_1^1 \geq 0, \quad d\rho_2^1 > 0, \quad d\phi_2^1 = 0,$$

$$d\rho_1^2 < 0, \quad d\phi_1^2 = \begin{cases} \geq 0 & \text{if } \rho_2 \leq 0 \\ \leq 0 & \text{if } \rho_2 \geq 0 \end{cases}, \quad d\rho_2^2 = 0, \quad d\phi_2^2 > 0.$$

So we get that:

$$\omega(\partial \xi^1, \partial \xi^2) = \begin{vmatrix} d\rho_1^1 & d\phi_1^1 \\ d\rho_2^1 & d\phi_2^1 \end{vmatrix} + \begin{vmatrix} d\rho_1^2 & d\phi_1^2 \\ d\rho_2^2 & d\phi_2^2 \end{vmatrix} > 0.$$

as desired, where $\partial \xi^i$ is the tangent vector to the $\xi^i$ curve.

We can clearly define an analogous embedding for the black triangles, with the same amoeba image on the $(\rho_1, \rho_2)$ projection, so that we get an smooth embedding of the surface as required.

**Remark 4.13** Note that we could simply take $\rho_1 \equiv 0$ over the surface, and it would still satisfy (4–2). Hence we can indeed get a symplectic embedding of our surface over a fixed interval in $B$.

Now notice that we chose our amoeba (with small $\delta$) so that the tangency of the amoeba at the vertices are far from being orthogonal to the corresponding $w_j$ direction. Name $H_j$ the half plane whose boundary line is normal to $w_j$, passes through $b$, and contains the vertex of the amoeba corresponding to $w_j$. The symplectic conditions (4–2) and (v) from Definition 4.6, ensures that the path in $\mathcal{C}(\Gamma)$ associated to $w_j$ is in $H_j$. We can then connect it to the corresponding $p_j$ via a path still satisfying the condition of never being orthogonal to $w_j$.

\[\square\]

### 4.3.3 Connecting the local models

Now the interior vertex local model gives us an $m_j$ copies of the boundary corresponding to $w_j$, equidistant in our parametrisation of $T^2$. Following the path that passes through $p_j$, as in Proposition 4.12: if it hits another interior vertex, we just follow this constant cycles and the surfaces glue naturally; if it hits a boundary vertex over the boundary of $\Delta$, building the surface again by keeping the cycles constant, the cycles smoothly collapses to a point since near the boundary of $\Delta$ we have a toric model; now to glue to the local model given by Proposition 4.10, we need to do some extra work, given the nature of the boundaries coming out of Proposition 4.10, recall Figure 9.

To connect $k$ straight cycles (we color red) to $k$ curling cycles (we color blue) – see again Figure 9 – we consider dimers as the ones illustrated in Figure 14 (for $k = 3$ and 4, but that clearly generalizes for any $k \geq 1$). Independent of $k$, the components containing the vertices of these dimer models are either a bi-gon, a 3-gon or a 4-gon. The analysis for getting a symplectic embedding for each of these pieces,
especially the 3-gons and the bi-gons, is similar to the one we made in the proof of Proposition 4.12, and we employ the analogous terminology now. Here the main difference is that the $\rho_2$ coordinate at most of the pieces of the blue cycles is not constant, but this was not crucial for getting the symplectic curve in Proposition 4.12. This makes the boundary of the surface project into segments, rather than points. To replicate the analysis in the proof of Proposition 4.12 to the 4-gons, it is better to subdivide it into two 3-gons and two bigons. Figure 15 illustrates the coamoebas (with some $\rho_2$ level sets depicted), the corresponding piece of the smooth surface and the amoeba should look like. Remark 4.13 still holds and we could actually view the amoeba picture as straight segments, but the blue boundaries would still project to sub-segments.

Now, we do not know that $\partial \sigma_j$’s, who live in the product of an interval with $T^2$, project exactly in the pattern as depicted in Figure 14, for $j = 2, \ldots, k$. The only information we have is that they are mutually linked and the projection intersect each horizontal cycle generically twice. But that is not a problem
since for each one of them we can draw the actual projection and the a corresponding generic desired curve as in Figure 14. One can see that they will generate a dimer, either one annulus or two bi-gons. So we can connect them with a smooth symplectic surface. The condition, $\Psi_k(s) > \Psi_{k-1}(s) > \cdots > \Psi_2(s)$ we required in the discussion before the statement of Proposition 4.10, ensures that we can place these surfaces inside mutually disjoint thickened tori for $j = 2, \ldots, k$. This explains how to patch the boundaries $\partial \sigma_j$’s to straight cycles for our $(\theta_1, \theta_2)$ coordinates on the torus.

Finally, the cycles built in Proposition 4.12 are equidistant in our coordinates for $T^2$, but the straight cycles we just built are thought to be very close to $\partial \sigma_1$ (In these sense, Figure 14 is misleading for visual purposes). Using the notation of Proposition 4.12, we promptly see that this is not a problem, since we can move apart these close cycles, though to be given by $\phi_2$ constant, until their projection to $T^2$ become equidistant, by moving only in the corresponding $\rho_2$ direction. The symplectic condition (4–2) is readily checked, by taking the first curve the horizontal ones in the coamoeba part, and the second curves vertical ones, so $d\rho_1^1 = d\rho_2^1 = 0$, $d\rho_1^2 = d\rho_2^2 > 0$, $d\phi_2^2 = 0$.

All these connecting surfaces can be made to project into tiny regions in the almost toric fibrations, in particular inside the neighborhood $N$ of our given symplectic-tropical curve $\mathcal{C}(\Gamma)$. We then finished the proof of Proposition 4.7.

From now on, we simplify notation calling $\mathcal{C}$ the symplectic-tropical curve obtained in Proposition 4.7 from the data $\mathcal{C} : \Gamma \rightarrow \Delta$.

4.3.4 Combinatorial background for triangular shaped ATFs

From [69], consider the ATBD of triangular shape for a del Pezzo surface containing the $\mathcal{C}^{n_1,n_2,n_3}_{p,q,r}$ monotone Lagrangian torus as a visible fibre, i.e., not inside a cut. This is related to the Markov type equation:

\[(4–3) \quad n_1 p^2 + n_2 q^2 + n_3 r^2 = G p q r \]

where $G = \sqrt{d n_1 n_2 n_3}$, $d$ is the degree of the corresponding del Pezzo. These yield the Diophantine equations in Subsection 3.2. In [69], it is shown that $n_1 p^2$, $n_2 q^2$ and $n_3 r^2$ correspond to the determinant of the primitive vectors associated with the corners of the corresponding ATBD.

Forcing the common edge of the corners corresponding to $n_2 q^2$ and $n_3 r^2$ to be horizontal, we get that the cuts and the primitive vectors of the remaining edges are as illustrated in Figure 16, where we
name \((l, q)\) and \((m, r)\) the direction of the cuts – compare with [65, Figure 13] and [68, Proposition 2.2, Figure 2]. The condition that the third determinant is \(n_1 p^2\) becomes:

\[
\begin{vmatrix} n_2 q l - 1 & n_3 r m + 1 \\ n_2 q^2 & n_3 r^2 \end{vmatrix} = n_2 n_3 r^2 q l - n_3 r^2 - n_2 n_3 q^2 r m - n_2 q^2 = n_1 p^2
\]

This yields the equalities

\[
n_2 n_3 r^2 q l - n_2 n_3 q^2 r m = n_3 r^2 + n_2 q^2 + n_1 p^2 = G p q r
\]

and upon dividing by \(n_2 n_3 q r\), we obtain

\[
rl - qm = \frac{G p}{n_2 n_3}
\]

Equation (4–5) is used to build symplectic-tropical curves in the \(\varepsilon\)-neighborhood depicted in Figure 16.

### 4.3.5 Symplectic-tropical curves in \(\mathcal{M}\)

Let \(\mathcal{B}\) be a neighborhood of an edge union associated cuts of an ATBD, as illustrated in Figure 16. Let \(\mathcal{N}\) denote the preimage of \(\mathcal{B}\) in \(X\). Looking for the combinatorics of the ATF, we show that we can construct a symplectic-tropical curve (Definition 4.6) inside \(\mathcal{B}\), and by Proposition 4.7, there is a corresponding symplectic curve in \(\mathcal{N}\). These symplectic curves will have the same intersection with the anti-canonical divisor as the rational curves highlighted in Figure 1. Their homology classes can differ from the rational curves listed in \(\mathcal{H}\) (Figure 1), by the classes of the Lagrangian spheres projecting in between two nodes inside \(\mathcal{N}\). In order to obtain them in the desired homology class, we will need to modify our curves in the pre-image of the neighborhood of the cuts, containing the Lagrangian spheres. This will be done in the next Subsection 4.3.6.

The collapsing cycle corresponding to the node associated with the cut \((l, q)\), respectively \((m, r)\), is represented by the orthogonal vector \((q, -l)\), respectively \((-r, m)\). Consistent with Definition 4.6 (vii), consider a symplectic-tropical curve with: one interior vertex; one leaf going towards one of the nodes with \((q, -l)\) collapsing cycle, with multiplicity \(r\); another going towards one of the nodes with \((-r, m)\) collapsing cycle, with multiplicity \(q\); and the third going towards the bottom edge, with multiplicity \(\frac{G p}{n_2 n_3}\).

They satisfy the balancing condition (4–1) of Definition 4.6 (ix), since from (4–5):

\[
r(q, -l) + q(-r, m) + \frac{G p}{n_2 n_3}(0, 1) = (0, 0).
\]

Hence we get a symplectic-tropical curve in \(\mathcal{N}\) by Proposition 4.7.

Let us understand the behaviour of a family of these curves as we deform our del Pezzo surface towards the corresponding limit orbifold. From the proof of Theorem 4.5 in [69], considering the limit orbifold and the orbiline \(\mathcal{A}\) corresponding to the limit of horizontal line in Figure 16, we get that the intersection of \(\mathcal{A}\) with the anticanonical divisor \([\mathcal{A}] + [\mathcal{B}] + [\mathcal{C}]\), where \([\mathcal{B}]\) and \([\mathcal{C}]\) are the classes of the other orbilines, is:

\[
[A] \cdot ([A] + [B] + [C]) = \frac{n_1 p^2 (n_3 r^2 + n_2 q^2 + n_1 p^2)}{n_1 n_2 n_3 p^2 q^2 r^2} = \frac{G p}{n_2 n_3 q r}
\]
Note that \( qr \) times this number is the one found in (4–5). Hence, the symplectic-tropical curve we construct limit to a orbi-curve in the class \( qrA \). In particular, for the symplectic-tropical curve to be a smoothing of that orbiline in the class \( A \), we must have \( q = 1 \) and \( r = 1 \). Indeed, by looking at the intersection with the Lagrangian spheres, we will see that the actual smoothing (in case \( qr = 1 \)) is given by a deformed symplectic-tropical curve.

4.3.6 Deforming symplectic-tropical curves

We need to be able to have a better control of the intersection number of our symplectic-tropical curve \( \mathcal{C} \) with the Lagrangian 2-spheres that appear naturally for a pair of nodes lying inside the same cut. Let us prove Proposition 4.14, with the following notation.

Let \( \mathcal{C} : \Gamma \rightarrow \Delta \) be a symplectic-tropical curve in a ATF of \( X \), represented by the ATBD \( \Delta \) and \( N \) a neighborhood of \( \mathcal{C}(\Gamma) \), as before. Consider a class of \( n \) nodes of the ATBD inside the same cut, so that at least one of the nodes are in \( \mathcal{C}(\Gamma) \) and let \( M \) be a neighborhood of the cut. Let \( S_1, \ldots, S_{n-1} \) be Lagrangian spheres projecting inside the cut to consecutive segments between the nodes. Also name \( S_0 = S_n = \emptyset \). Let \( m \) be the sum of the multiplicities of the leaves arriving at these \( n \) nodes, and choose \( d < n \) nodes, where \( m \equiv d \mod n \).

Proposition 4.14 There is a symplectic curve projecting into \( N \cup M \) with the property that its intersection with \( S_1 \cup \cdots \cup S_{n-1} \) is precisely the \( d \) chosen nodes.

The idea here is that \( m - d = kn \), so we place \( \sigma_j \) disks for Proposition 4.10, for \( j = 2, \ldots, k+1 \) at all the \( n \) nodes. Naming \( n_1, \ldots, n_n \) be the \( n \) nodes in a cut, in order, and \( S_i \) a Lagrangian sphere projecting to the cut between the consecutive nodes \( n_i, n_{i+1} \), we see that the signed intersection of \( S_i \) with the collection of \( kn \) disks is zero, since the \( S_i \) sphere only intersects the \( k \) disks around the \( n_i \) and \( n_{i+1} \) nodes, with opposite signs. So it is clear that at least smoothly we can pairwise cancel these intersections, but before we need to connect the disks.

Let us focus now on how to construct a symplectic pair o pants \( \mathcal{P} \) to glue to the boundary of a pair of \( \sigma_j \) disks near the \( n_i \) and \( n_{i+1} \) nodes.

![Figure 17: Amoebas of the surfaces \( \Sigma_{j,i}^{i+1} \), \( \Sigma_{j,i}^{i+1} \).](image)

Lemma 4.15 There is a symplectic disk \( \Sigma_{j,i}^{i+1} \) in \( M \), containing the \( \sigma_j^{i} \) and \( \sigma_j^{i+1} \) disks as a subset, with boundary in a thickened 2-torus \( \mathbb{T} \) in the class of two times the collapsing class of the nodes via the identification \( H_1(\mathbb{T}; \mathbb{Z}) \cong H_1(\mathbb{T}^2; \mathbb{Z}) \). We name \( \mathcal{P} \) the pair of pants which is the closure of \( \Sigma_{j,i}^{i+1} \setminus \sigma_j^{i} \cup \sigma_j^{i+1} \).

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Figure 17 shows on the left the amoeba corresponding to the $\Sigma_j^{i,i+1}$ surface. The idea is to then isotope it to a different surface $\Sigma_j'^{i,i+1}$ with boundary still in $IT$, such that the intersection with $S_i$ is empty.

Figure 18: (a) A dimer model for the symplectic pairs of pants $\mathcal{P}$. (b) Dimer model for $\mathcal{P}'$, illustrating the cycles in the $S_i$ Lagrangian sphere, over the intersection between the boundary of the amoeba of $\mathcal{P}'$ and the amoeba of $S_i$ (a segment between the nodes). (c) The dimer model for the part of the surface $\Sigma_j'^{i,i+3}$ connecting the boundary of the $\sigma_j$ disk in the $n_i+3$ node with the boundary of $\Sigma_j'^{i,i+2}$. (d) Coamoeba of 2 non-intersecting $\mathcal{P}'$s. The suggested height is with respect to the $p_2$ coordinate.

**Proof**  Let us assume the cut is vertical, so the collapsing cycles is given by a $\theta_2$ constant curve in the $(\theta_1, \theta_2)$ coordinates of $T^2$. Making a change of action-angles coordinates, if needed, we may assume that the collapsing cycles corresponding to the $n_i$ nodes are slightly fased out. So we draw the $T^2$ projection of the boundary of the $\sigma_j$ disks, recalling that it links once the horizontal cycle in a thickened torus that they live in (recall construction before Proposition 4.10). We color these boundaries blue, and draw a dimer model that indicates the coamoeba of the pair of pants $\mathcal{P}$ we will construct, as illustrated in the first picture of Figure 18. We will color the other boundary of $\mathcal{P}$ red. We number the components of the red curve in the dimer model of the leftmost picture of Figure 18, as indicated and sketch the profile of its $p_2$ coordinate as indicated in Figure 19.

The $\partial \sigma_j$ curves can be taken extremely close to the collapsing cycle $\partial \sigma_1$, so we assume its $\theta_2$ variation is small enough with respect to the difference $p_1$ between the coordinate of the red curve and the $p_1$ coordinate of the blue curve. So essentially we take them small with respect to the size of $\mathcal{M}$.

Figure 19: The profile of the $p_2$ coordinate as we move along $\partial \Sigma_j^{i,i+1}$, $\partial \Sigma'^{i,i+1}$. The numbering corresponding to the ones on the leftmost picture of Figure 18.

So the dimer model we are considering consists of two bi-gons and two tri-gons. We carefully analyse the $(p_1, p_2)$ coordinates we associate to these pieces, to ensure de get a symplectic pair of pants $\mathcal{P}$ connecting the blue boundaries to the red one. The corresponding amoebas, together with the $\theta_1$ and $\theta_2$ level sets are indicated in the middle of Figure 20. There is a curve in the amoeba corresponding to the vertices of the bi-gon, that will by crucial in some further analysis. We name it the pink curve, and in this case it is a horizontal segment.
Following analogous convention for the analysis of amoebas and coamoebas as in the proof of Proposition 4.12, with \( \rho_i \) corresponding to \( p_i \) and \( \phi_i \) to \( \theta_i \), we take the first curve \( \xi^1 \) corresponding to a \( \theta_2 \) level set, oriented with \( d\theta_1 > 0 \) and the second curve \( \xi^2 \) corresponding a \( \theta_1 \) level set, with the orientation such that the \( p_1 \) coordinate is decreasing, i.e., \( dp_1^2 < 0 \).

Let us look now to the bi-gon. We assume that the \( \theta_2 \) variation is also very small for the red curve as well, so if we fix the norm of \( dp_2^2 \) as we travel, we then have \( d\theta_2 \) is very small away from the vertices. As indicated in Figure 20, we chose \( \xi^1 \) so that \( dp_1^2 = 0 \) and \( d\theta_2 = 0 \). As we move along \( \xi^1 \) at the bottom of the bi-gon, the \( p_1 \) coordinate increases up to the middle of the horizontal curve at coamoeba and decreases from the middle until the end, it reverses the logic as we move to the top of the bi-gon. So, \( \omega(\partial \xi^1, \partial \xi^2) = -d\theta_1^1 dp_1^1 > 0 \) (recall equation (4–2)), as desired. The limit case where \( \theta_2 \) obtains the maximum and the minimum in the bi-gon can be analysed by replacing the \( \xi^1 \) curve by the respective red and blue curves. Since it also corresponds to maximum and minimum of the \( p_2 \) coordinate, we still have \( dp_1^2 = 0 \) and \( d\theta_2^2 = 0 \), ensures positivity of \( \omega(\partial \xi^1, \partial \xi^2) \).

We now turn our attention to the tri-gon, as depicted in Figure 20. We first notice that the \( p_2 \) coordinates associated with the bottom vertex of the tri-gon, must be different for the edge labeled 2 and the edge labeled 6, the later being bigger than the former, recall Figures 18 and 19. This implies that we must not only consider one curve with that corresponding \( \theta_2 \) coordinate constant, but rather a family of curves. The shaded region in the bottom-middle picture of Figure 20 indicates the image of this curves under the \( (p_1, p_2) \)-projection. The analysis at the top of this part of the coamoeba is similar to before, with \( dp_2^2 = 0 \) and \( d\theta_2^2 = 0 \) and \( \omega(\partial \xi^1, \partial \xi^2) = -d\theta_1^1 dp_1^1 > 0 \) (we don’t remain stationary with \( d\theta_1^1 = 0 \) at the top part of the coamoeba). At the bottom part, we do have \( dp_1^2 > 0 \) as \( \xi^1 \) travels from the 2 curve to the 6 curve and we also have \( d\theta_2^2 > 0 \), as we are traveling from the red curve to the blue curve along \( \xi^2 \), with \( dp_2^2 < 0 \). That way, since \( d\theta_1^1 \geq 0 \), we have
\[ \omega(\partial \xi^1, \partial \xi^2) = -d\theta_1^2 dp_1^2 + d\theta_2^2 dp_2^1 > 0. \]

Gluing analogous models for the for the bottom part of the diagram in a consistent way provides our symplectic pair of pants.

We now move towards the construction of the surface \( \Sigma_{j}^{n,j+1} \) that does not intersect the Lagrangian sphere \( S_i \). Given our choice of coordinates, the Lagrangian sphere projects into a vertical segment in the \((p_1, p_2)\) factor. In order to keep working with the same coordinates in the region we want to do the modification of our curve, we modify the ATBD, without changing our ATF, simply by reversing the direction of the cuts associated with the nodes \( n_{i+1}, \ldots, n_n \) (this operation was named \textit{transferring the cut} in [68]). Note that we already made this operation in Figures 17, 20. In Figure the rightmost diagrams of 20 we represent the projection of \( S_i \) in a consistent way provides our symplectic pair of pants.

From Remark 4.9, we see that \( \sigma_j^{i} \) and \( \sigma_{j+1}^{i} \) both intersect \( S_i \) once, and one can check it is with opposite signs. We consider disks \( \sigma_j^{i'} \) and \( \sigma_{j+1}^{i+1} \), obtaining from \( \sigma_j^{i} \) and \( \sigma_{j+1}^{i} \) by carving out that contains the intersection point with \( S_i \), with their \((p_1, p_2)\) projection as illustrated in the second picture of Figure 17. We keep colouring their boundary blue, and their coamoeba projection is similar to the ones we just analyse, can very close to the corresponding collapsing cycle with constant \( \theta_2 \) coordinate. So we can also build a dimer model as in the leftmost picture of Figure 18, to build a pair of paints \( P' \), and we also color the other boundary of \( P' \) red. Recall that we named a pink curve, the \((p_1, p_2)\) image of the vertices corresponding to the intersection of the blue and red curves in the coamoeba coordinates \((\theta_1, \theta_2)\), which we will refer to as pink vertices. It will play the following crucial role in our construction. As before, we focus on the top part of the coamoeba, the bottom part being symmetric under the reflection around the \( p_2 = 0 \) coordinate (or at least having a symmetric behaviour, since our coamoeba picture is not symmetric). The pink curve corresponding to the top bi-gon will be the graph of a non-increasing convex function \( p_2(p_1) \), starting at a point with \( p_1 < 0, p_2 > 0 \), becoming negative before \( p_1 \) becomes 0, and eventually becoming constant at some point where \( p_1 > 0 \). Hence the pink curve for the bottom bi-gon will intersect this at some point with negative \( p_1 \) coordinate. We call \( x \) the endpoint in the \( p_2 \) constant segment, with smallest \( p_1 \) coordinate. For that to happen, the \( p_2 \) coordinate of the red curve needs to move different as we move along the different parts of the coamoeba, as indicated in Figure 19. So it maintains the property that the \( p_2 \) coordinate at the common point of the segments labeled 2 and 3 is smaller than the one corresponding to the 5 and 6 segments. We also need to ensure the following. Consider the point of intersection between the \((p_1, p_2)\) projection of the top blue curve and the projection of \( S_i \). Look at the \( \theta_2 \) coordinate of the circle of \( S_i \) over this point, and consider its intersections \( q_1, q_2 \) with the blue curve. We draw the coamoeba profile of the red curve so that \( q_1, q_2 \) are precisely the pink vertices, as illustrated in the middle picture of Figure 18. We make analogous choices for drawing the red curves at the bottom of the coamoeba.

We are now in shape to prove:

**Lemma 4.16** There is a symplectic disk \( \Sigma_{j}^{n,j+1} \) in \( \mathcal{M} \), disjoint from \( S_i \), containing the \( \sigma_j^{i} \) and \( \sigma_{j+1}^{i} \) disks as a subset, with boundary in a thickened torus \( \mathbb{T} \) in the class of two times the collapsing class of the nodes via the identification \( H_1(\mathbb{T}; \mathbb{Z}) \cong H_1(\mathbb{T}^2; \mathbb{Z}) \). We name \( P' \) the pair of pants which is the closure of \( \Sigma_{j}^{n,j+1} \setminus \sigma_j^{i} \cup \sigma_{j+1}^{i} \).
**Proof** As in the previous proof, we will draw in the \((p_1, p_2)\) coordinates the level sets of the \(\theta_2\) and \(\theta_1\) coordinates, naming the former \(\xi^1\) and the later \(\xi^2\). Let us start looking at the bi-gon, and describe the projection of the \(\xi^1\) curve in the amoeba. Each curve starting close to the \(\theta_2\) minimum, up to a certain height \(b\) (to be specified), will have a horizontal projection, with the maximum of the \(p_1\) coordinate corresponding to the half of the \(\theta_1\) coordinate of \(\xi^1\). This last property remains the same, even when we start at a height bigger than \(b\), but then, the image becomes a graph of a non-increasing function \(p_2(p_1)\), eventually limiting to the part of the pink curve that stops at \(x\), as illustrated in the top-right picture of Figure 20. We assume that the derivative is smaller in norm to the derivative of the graph giving the pink curve. For the top part of the coamoeba, the \(\xi^1\) curves have constant \(p_2\) coordinates, with the minimum of the \(p_1\) coordinate happening at the middle of the \(\theta_1\) coordinate. The amoeba projection of the \(\xi^2\) curves are also indicated in top-right picture of Figure 20. In particular, analysing the symplectic condition for the top of the coamoeba part, is essentially the same as done in the proof of Lemma 4.15, compare top-middle and top-right pictures of Figure 20. To ensure the symplectic condition at the bottom of the coamoeba, we need to carefully choose the point \(b\), recalling that we can choose the red curve so that the \(\theta_2\) variation is very small compared with the \(\theta_1\) variation. Away form the points in \(\xi^1\) with maximum \(p_1\) coordinate, let us move in the \(\xi^1\) curves with the normalised condition of \(|dp_1^1| = 1\). The variation \(d\theta_2^2\) then will be bounded by an extremely small constant (w.r.t. the \(\theta_1\) diameter the coamoeba). We take a constant \(b\), high enough to ensure that \(\omega(\partial \xi^1, \partial \xi^2) = d\theta_1^1 - dp_1^1 d\theta_2^2 > 0\), recalling that we forced \(dp_1^1\) to be zero for points at \(\theta_2\) heights smaller than \(b\) and \(|dp_2^2|\) is bounded by the maximum slope of the pink curve. In the points in \(\xi^1\) with maximum \(p_1\) coordinate, we simply have \(dp_2^1 = 0\) and \(\omega(\partial \xi^1, \partial \xi^2) = -d\theta_1^1 dp_2^1 > 0\). We let the reader check the positivity for the limiting points at the pink curve.

Let us now move to the tri-gon part of the coamoeba. For the top part we choose a height \(c\) in the coamoeba, playing a similar role as \(b\), in the last paragraph. If the \(\theta_2\) coordinate of \(\xi^1\) is bigger than \(c\), we take the amoeba part to have constant \(p_2\) coordinate. So if the \(\theta_2\) coordinate is not smaller than \(c\), we have \(dp_1^1 = 0\) and \(\omega(\partial \xi^1, \partial \xi^2) = -d\theta_1^1 dp_2^1 > 0\). The analysis regarding the symplectic condition is done as in the last paragraph for the part corresponding to the \(\theta_2\) coordinate smaller than \(c\). For the bottom part, recall that the \(p_2\) coordinate corresponding to the part 2 of the coamoeba of the red curve is smaller than the one corresponding to the part 6. So we can take \(dp_2^1 > 0\). The analysis now is similar to the analogous part in the proof of Lemma 4.15. We have \(d\theta_1^1 \geq 0\) (being zero at the middle of the \(\xi^1\) curves whose \(\theta_2\) coordinate is not bigger than the pink vertices), \(dp_1^1 > 0, d\theta_2^1 = 0, dp_2^1 < 0, d\theta_1^1 = 0, d\theta_2^2 > 0\), since we move on \(\xi^2\) from the red curve to the blue curve.

The fact that we chose the pink curve to cross \(p_1 = 0\) with negative \(p_2\) coordinate promptly ensures that the coamoeba region of \(S_i\) corresponding to the segment given by the intersection of \(p_1 = 0\) with amoeba of the bi-gon, do not intersect the bi-gon itself. For the tri-gon part, looking at the amoeba projection, we see that \(S_i\) does not intersect the region of the surface corresponding to the bottom of the tri-gon. The fact that we chose the coamoeba of our red curve so that the pink vertices coincide with the points \(q_1, q_2\), ensures that the part of the surface whose coamoeba correspond to top of the tri-gon does not intersect the Lagrangian \(S_i\) as well.

\[ \square \]

**Remark 4.17** There is a smooth way to isotope the boundary of \(\mathcal{P}\) to the boundary of \(\mathcal{P}'\), and following that, a smooth way to isotope the amoebas of \(\Sigma_{j,i+1}^j\) to the ones of \(\Sigma_{j,i+1}^j\), see again Figure 20, as well as the small differences on the coamoebas. Hence, we can isotope from \(\Sigma_{j,i+1}^j\) to \(\Sigma_{j,i+1}^j\), with their boundaries restricted to a thickened torus \(T = I \times T^2\). \[ \square \]
Now we see that \( \Sigma^i_{j,i+1} \) intersects \( S_{i+1} \) once in a point belonging to \( \sigma''_{i+1} \). In an analogous fashion to Lemma 4.16, we can chop out of \( \Sigma^i_{j,i+1} \) the intersection point with \( S_{i+1} \) and then glue its boundary, the boundary of \( \sigma''_{j+2} \) (where we remove the intersection of \( \sigma''_{j+2} \) with \( S_{i+1} \)), and the boundary of a new chosen "red curve" in a thickened torus that has three times the homology of the collapsing cycle in this thickened torus, with a new pair of pants. Naming the former two curves blue, the first step would be to construct a dimer model between the blue curves and the red curve, so that the behaviour on the region that it could intersect \( S_{i+1} \) is the same as the one analysed in Lemma 4.16. We name this surface \( \Sigma^i_{j,i+2} \). We can keep interacting and consider symplectic surfaces \( \Sigma^i_{j,i+k} \) in \( M \), that do not intersect \( S_i, \ldots, S_{i+k-1} \), and has boundary on a thickened torus in the class \( k+1 \) times the collapsing cycle. We eventually get:

**Lemma 4.18** There exists a symplectic disk \( \Sigma^i_{1,n} \) inside \( M \), that has a boundary on a thickened torus \( I \times T^2 \), in the class \( n \) times the class of the collapsing cycle, that does not intersect \( \bigcup_{i=1}^{n-1} S_i \).

**Proof** We build it up interactively as indicated above, starting with \( \Sigma^i_{1,2} \). The algorithm to build the red curve and the corresponding dimer model is illustrated for going from \( \Sigma^i_{1,3} \cup \sigma^i_j \) to \( \Sigma^i_{1,4} \) in the third diagram of Figure 18. We chose the red curve to cross in consecutive chambers of the complement of the curves given by the amoeba projection of the blue curves, which we recall is the boundary of the disconnected surface obtained by chopping out the intersections of \( \Sigma^i_{1,j} \cup \sigma^i_{j+1} \) with \( S_j \). We do it so that the top part of the dimer, corresponding to one bi-gon and two tri-gons have the same configuration as in Lemma 4.16, including the points analogous to \( q_1, q_2 \). Recalling Remark 4.17, we can think that we first glue a pair of pants \( P \) as before for the boundaries of \( \Sigma^i_{1,j} \cup \sigma^i_{j+1} \) and then isotope to our desired surface \( \Sigma^i_{1,j+1} \), with the modifications happening in the same framework as in the proof of Lemma 4.16.

![Figure 21: Final amoeba for the symplectic disk in the case of a symplectic-tropical curve arriving at \( n = 4 \) nodes with multiplicity \( m = 5 + 6 = 11 \), so \( k = 2 \) and \( d = 3 \). For visual purposes, we illustrate the amoeba of \( k = 2 \) surfaces as if one envelopes the other, when in fact both have the same amoebas.](image)

Now we should inductively build the surfaces \( \Sigma^i_{j,n} \), for \( j \) going from 1 to \( k \) making sure these surfaces do not intersect. Recall that the boundary of \( \sigma_j \) and \( \sigma_{j+1} \) are linked. We can achieve non-intersection by adjusting the crossings of blue and red curves between different amoebas, and the \( p_2 \) coordinate. So we can ensure non intersection just by looking at the \( (\theta_1, \theta_2, p_2) \) projection of the surfaces. The case \( n = 2 \) is illustrated in Figure 18 (d). We can then get the other \( d \) disks, carrying the collapsing cycle from the respective nodes to boundary of the same thickened torus. They project to curves inside the amoeba of
the $\Sigma_j^{1,n}$ disks, see Figure 21. After doing that, still within $\mathcal{M}$ we can connect the boundaries of the $\Sigma_j^{1,n}$ surfaces, $j = 1, \ldots, k$ and of the $d$ disks into $m = kn + d$ straight cycles and redistribute these cycles over curves connecting to our symplectic-tropical curve $\mathcal{C}(\Gamma)$, as we did in the Section 4.3.3. Figure 21 illustrates the amoeba image of this local model of a deformed symplectic-tropical curve, when $n = 4$, $m = 11$, so $k = 2$ and $d = 3$. This finishes the proof of Proposition 4.14.

We can glue all these local models now, as we did in Section 4.3.3 to get smooth deformed symplectic-tropical curves, intersecting Lagrangian spheres only at prescribed nodes, with the number determined by the total multiplicity and the number of nodes at a given cut.

4.4 Required Symplectic Tropical Curves in $\mathcal{N}$

Consider the triangular-shaped ATFs of $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $Bl_3(\mathbb{C}P^2)$ and $Bl_4(\mathbb{C}P^2)$, [69] with one smooth corner, i.e., satisfying a Markov type equation of the form

\[(4–8) \quad 1 + n_2q^2 + n_3r^2 = Gqr.\]

Let $\mathcal{N}$ be a neighborhood of the edge opposite the frozen smooth vertex, introduced in Section 4.3.5. The results of the previous subsections yield the following

**Theorem 4.19** Each homology class of the symplectic divisors highlighted in Figure 1, can be realised as a symplectic-tropical curve in $\mathcal{N}$.

**Proof** First note that for the equations associated with $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $Bl_3(\mathbb{C}P^2)$ and $Bl_4(\mathbb{C}P^2)$, $G/n_2n_3$ is respectively, 3, 2, 1, 1, which is the intersection number of the symplectic divisors in the corresponding spaces with the anti-canonical divisor. So Proposition 4.7 and Section 4.3.5 completely nail down the case of $\mathbb{C}P^2$. For the remaining cases, one needs to ensure the correct intersection with the Lagrangian spheres. This can be achieved case by case, using that $1 + n_2q^2 \equiv 0 \mod n_3$ and $1 + n_3r^2 \equiv 0 \mod n_2$. For instance, in the case of $Bl_4(\mathbb{C}P^2)$, we get $q^2 \equiv -1 \mod 5$, so $q \equiv 2$, or 3 mod 5. Take the divisor of Figure 1 that intersects two Lagrangian spheres. Following the mutations in [69, Figure 17], at the triangular-shape ATBD [69, Figure 17 ($A_4$)], these spheres become the top and bottom spheres, of the 4-chain of Lagrangian spheres. If $q \equiv 2 \mod 5$, apply Proposition 4.14 taking the 2 intersections to the top and bottom node. If $q \equiv 3 \mod 5$, we arrive at the three interior nodes instead, having the same intersection number with the Lagrangian spheres. One can check that the sign is correct by looking at the first instance when $q = 2$ and look that the mutations $q \to 5r - q$ switches between $q \equiv 2 \mod 5$ and $q \equiv 3 \mod 5$ (the first case $q = 2$ is depicted as $A_1$ in Figure 23).

4.5 Further Deformations of Symplectic Tropical Curves

In this Section we need to perform additional tricks regarding symplectic-tropical curves, that will allow us to visualize chains of them inside an ATF. When we say a chain of symplectic curves, we imply that the total intersection between them is equal to the topological intersection. So it is not enough to simply construct, for each curve in the chain, as STC, as we did in the previous sections. We start with two simple observations:
Remark 4.20  First, for \( i = 1, 2 \), let \( \mathcal{C}_i \) denote two STCs as in Definition 4.6. Let \( C_i \) denote a STC in \( X \) represented by \( \mathcal{C}_i \) as in Proposition 4.7. Let \( \gamma_i \) be an edge of \( \mathcal{C}_i \). If \( \gamma_1 \) and \( \gamma_2 \) homology class of the torus, any intersection between \( \mathcal{C}_1(\gamma_1) \) and \( \mathcal{C}_2(\gamma_2) \), can be taken to be empty as intersection of \( C_1 \) and \( C_2 \), since we can just assume we carry disjoint cycles in the same homology class.  

Remark 4.21  Second, consider the part of a symplectic-tropical curve disjoint from a neighborhood of the cuts. We can get a parallel symplectic-tropical curve, with the same amoeba, by simply shifting, say the \( \theta_2 \) coordinate of the coamoeba, which is well-defined in the complement of the cuts.

Figure 22: (a) describe the amoeba of a STC given by a \( \sigma_j \) disk (from Proposition 4.10), glued along the boundary with two cycles intersecting once \( \partial \sigma_j \); (b) describes the coamoeba of the gluing surface; (c) illustrates how the surface looks like inside a thickened torus.

We will also need the following:

Proposition 4.22  Consider a thickened torus \( IT = [-\epsilon, \epsilon] \times T^2 \), as the pre-image of a segment in the regular part of a base of an ATF. Let \( \alpha \) be a straight cycle in \( \{0\} \times T^2 \) represented by \( v \in H_1(IT; \mathbb{Z}) \cong H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2 \). Let \( \beta \) by a cycle in the class \( v \), that wraps around once \( \alpha \). Let \( \gamma_\pm \) be a straight cycle in \( \{\pm \epsilon\} \times T^2 \) represented by \( u_\pm \), with \( \det u \wedge v = \pm 1 \), and \( u_+ = u_- - v \). Then, in \( IT \setminus \alpha \), there is a symplectic pair of pants with boundary \( \gamma_- \cup \beta \cup \gamma_+ \).

Proof  Use the dimer model represented in Figure 22 (b), to build a symplectic surface as in Section 4.3.2, making sure that the 0 level set of the height function (\( \rho_2 \) in Proposition 4.12) is disjoint from the straight cycle \( \alpha \). The end result is depicted in Figure 22 (c).

Remark 4.23  Applying Proposition for \( \partial \sigma_j \), the boundary of a disk \( \sigma_j \) as in Proposition 4.10, we see that we can pass with all \( \sigma_l \), with \( l < j \) (the ones with boundary closer to \( \alpha = \partial \sigma_1 \)) through the middle of the surface constructed in Proposition 4.22. The amoeba of this process is depicted in Figure 22 (a).

In Figure 23, we apply the above Remark 4.23 to visualise STCs in the neighborhood \( \mathcal{N} \) of highlighted edge of the ATF of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) depicted in the top corner of Figure 1 They are representative of the classes \( H_1 = [\mathbb{C}P^1 \times \{\text{point}\}] \) and \( H_2 = [\{\text{point}\} \times \mathbb{C}P^1] \). Remarks 4.23, 4.20, are used to visualize a 4-chain of STCs in the neighborhood \( \mathcal{N} \) of highlighted edge of the ATF of \( Bl_3(\mathbb{C}P^2) \) of triangular shape depicted in Figure 1. The homology classes for the spheres in this 4-chain, are the ones corresponding to the highlighted edges of the toric diagram for \( Bl_3(\mathbb{C}P^2) \) in Figure 1. In Figure 23 also show a 3-chain of symplectic spheres in the ATBD of \( Bl_4(\mathbb{C}P^2) \) of \([69, \text{Diagram (A4)}]\). Their classes corresponds to the highlighted 3-chain in the first diagram of \( Bl_4(\mathbb{C}P^2) \) in Figure 1.
Now, we can interactively apply Remark 4.23, in the neighborhood of one or more nodes. We are going to use simplified pictures, for visual purposes. For instance, Figure 24 (a) shows a simplified depiction of two nonintersecting STCs near two nodes of a cut in an ATF, with associated vector $v$. Surrounding each node, we see a $\sigma_j$ type curves, where we interacted 4.23 a times around each node and then unite their amoebas using Remark 4.21. Figure 24 (b), show an alternative version, where we took 2b curves around a unique node in associated with the vector $w$, and applied Remarks 4.23, 4.20, 4.21 to get two disjoint symplectic curves.
4.6 Getting chains of symplectic-tropical curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$

Consider a triangular-shaped ATF of $\mathbb{CP}^1 \times \mathbb{CP}^1$, with a smooth corner, associated to the Diophantine equations

$$1 + q^2 + 2r^2 = 4qr,$$

and $\mathcal{N}$ a neighborhood of the edge opposite the smooth corner, together with the associated cuts, as in Section 4.3.5.

**Proposition 4.24** There is a 2-chain of symplectic-tropical curves inside $\mathcal{N}$, with the curves in classes $H_1$ and $H_2$, with the total intersection one, i.e., equals to the topological intersection.

**Proof** We need to revisit the specific combinatorics of the situation. Recall that the associated Markov type equation of interest is $1 + q^2 + 2r^2 = 4qr$, and from (4–5), we see that the determinant between associated vectors $v = (-r, m)$ and $w = (q, -l)$ has absolute value 2. Moreover, one can see from the corresponding Vieta jumping (Proposition 3.9) that $q = 2a + 1$ and $r = 2b + 1$ are odd.

So we can rewrite the balancing condition as:

$$(4–9) \quad qv + rm + 2(0, 1) = v + m + 2u = 0$$

where $u = av + bw + (0, 1)$. We readily see that $|u \wedge w| = |u \wedge v| = 1$, So we are allowed to use the vectors $\pm u$, with each of the vectors $v$ and $w$ as in Proposition 4.22. Finally, we look at

$$u \wedge (0, 1) = -\frac{q-1}{2}r + \frac{r-1}{2}q = \frac{r-q}{2}.$$

Assume that $q > r$, so that $u$ points to the left. We get that $2v - u \wedge (0, 1) = \frac{q-5r}{2} < 0$, because $1 + 2r^2 = q(4r - q)$, so $q < 4r$.

Figure 25: STCs in classes $H_1$, $H_2$ in a neighborhood $\mathcal{N}$ of the edge opposite to the smooth corner in a arbitrary ATF of $\mathbb{CP}^1 \times \mathbb{CP}^1$ associated with a solution of $1 + q^2 + 2r^2 = 4qr$.

Figure 25 then is a depiction of the chain of symplectic spheres in $\mathcal{N}$ intersecting once, by Remarks 4.20, 4.21, 4.23, and the balancing condition (4–9) in the case $q > r$.

In the case $r < q$, we replace the vector $u$ by $-u$ in Figure 25, and note that $u \wedge (0, 1) < 0$ and $2v + u \wedge (0, 1) = \frac{-q-5r}{2} < 0$, as well. The case $r = q$ is depicted in Figure 23.

Chains of symplectic tropical curves in $Bl_3(\mathbb{CP}^2)$ and $Bl_4(\mathbb{CP}^2)$ should be constructed with the same techniques. Since the combinatorics of each of the Diophantine equations in Subsection 3.2 lead to prescribed intersection, one must adapt the numerics accordingly.
5 Quiver Combinatorics and Ellipsoid Embeddings

The aim of this section is to discuss the connection between the numerics of sharp ellipsoid embeddings and the arithmetic of cluster algebras. Let $M \cong \mathbb{Z}^2$ be the character lattice of the algebraic torus $(\mathbb{C}^*)^2$, and $N = \text{Hom}(M, \mathbb{Z})$ its dual lattice. Let us fix a 4-dimensional symplectic toric variety $X_\Delta = (X, \omega)$ with fan $\Delta \subseteq N$, up to $\text{GL}_2(\mathbb{Z})$-equivalence. Following the strategy presented in this article, the first question we are addressing is the existence of a sharp symplectic embedding

$$i : E(1, a) \rightarrow (X, \omega),$$

for values of $a \in \mathbb{R}^+$. For the cases in which the symplectic embedding $i$ exists, the second question is addressing the existence of a symplectic embedding

$$i : E(1, a) \rightarrow (X \setminus D, \omega), \quad a \in \mathbb{R}^+,$$

where $D \subseteq (X, \omega)$ is a symplectic divisor. This second part can be solved as in Subsection 3.3, and the arithmetic coincides for the sequence of absolute sharp embeddings $\{i_n\}$ and its relative $\{\iota_n\}$ counterpart. Thus onwards, only the problem of ellipsoid embeddings into closed toric varieties is addressed. Let us now provide an algebraic recipe for potentially finding convergent sequences $\{\alpha_n\}_n \in \mathbb{N}$ of positive real numbers such that $E(1, \alpha_n)$ sharply embeds into $(X, \omega)$, constructing sharp points for an infinite staircase.

For that, let us introduce the cluster algebra $C_\Delta$ associated to the fan $\Delta \subseteq N$ [1, 39]. Let $E = |E(\Delta)|$ be the number of edges of the fan $\Delta$. The initial seed for $C_\Delta$ is given by the following quiver $Q_\Delta$:

1. The vertices of $Q_\Delta$ are associated to edges $\{e_1, \ldots, e_E\}$ of the fan $\Delta$.
2. The number of arrows between to vertices $v_i, v_j$, associated to edges $e_i, e_j$ is given by

$$\max \{\rho_i \land \rho_j, 0\}$$

where $\rho_i$ is the primitive normal vector of the edge $e_i$, $1 \leq i, j \leq E$.

Thus, the cluster variables for this seed are indexed by the edges $\{e_1, \ldots, e_E\}$ of the fan, and the skew-symmetric exchange matrix $B$ is given by the exterior products of the two primitive normal vectors associated to each pair of edges. Cluster mutation consists of quiver mutation in pair with a mutation of the cluster variables. By definition, the cluster algebra $C_\Delta$ associated to $\Delta$ is generated by all the cluster variables of all the seeds obtained by mutating our initial seed [20].

5.1 The algebraic recipe

The fan associated to an ellipsoid, or rather its compactification to a weighted projective space, is triangular. Given that the problem we address is that of ellipsoid embeddings, the first ingredient in our recipe is a sequence $\mu_1, \ldots, \mu_s$ of mutations which brings $\Delta$ to a triangular fan $\Delta_0$. These mutations are specified by an ordered choice of possibly repeated vertices $v_1, \ldots, v_s$, $v_i \in V(Q)$, $1 \leq i \leq s$, in the quiver. Given that the fan $\mu_\ell(\ldots(\mu_1(\Delta))$) is triangular, the sequence of mutations $\mu_1, \ldots, \mu_s$ specifies a partition of $V(Q)$ into three sets $V(Q) = T_1 \cup T_2 \cup F$, where $T_1, T_2$ are determined by which edge of $\mu_\ell(\ldots(\mu_1(\Delta))$ they are sent to and the set $F$ consists of the set of vertices which are not mutated at in the list $\mu_1, \ldots, \mu_s$.

Given the quiver $Q$, we declare the subset $F \subseteq V(Q_\Delta)$ to be frozen vertices, i.e. coefficient variables for the associated cluster algebra [20]. Evaluate the cluster variables in $T_1 \cup T_2$ to their initial value for
a minimal Markov triple \((1, b, c)\), \(b, c \in \mathbb{N}\), where the cluster variables for \(T_1\) are set to \(b\) and those of \(T_2\) to \(c\). The recipe now starts from \(\mu_s(\ldots(\mu_1(Q)))\) and reads as follows:

(i) Mutate at all the vertices \(v \in T_1\),
(ii) Mutate at all the vertices \(v \in T_2\),
(iii) Repeat Steps (i) and (ii).

First, it is relevant to observe that the order in which you mutate the vertices in a given set \(T_1\), or \(T_2\), does not matter. This is algebraically interesting, given that these vertices are not necessarily disjoint, but geometrically clear.

Second, at the \(k\)th iteration of the recipe, \(k \in \mathbb{N}\), the cluster variables associated to the vertices \(v \in T_1\) have all the same value \(T_1(k)\), and the cluster variables associated to the vertices \(v \in T_2\) have all the same value \(T_2(k)\). The required sequence is given by

\[
a_n = T_1 \left( \frac{n + 1}{2} \right), \quad \text{if } n \text{ odd}, \quad a_n = T_2 \left( \frac{n}{2} \right), \quad \text{if } n \text{ even},
\]

and the potential infinite staircase would have sharp ellipsoid embeddings points at \(\alpha_n\) being the appropriate Markov successive quotients of \(\{a_n\}\). This yields a cluster algebra perspective to the numerics in Subsection 3.2. Let us give the necessary detail for the almost-toric basis in the second row of Figure 1.

Example 5.1 Let \((X, \omega) = X(\mathcal{P}) = Bl_3(\mathbb{C}P^2)\) be the monotone delPezzo surface \(dP_3\), corresponding to the symplectic (almost)-toric 4-manifold represented by the leftmost almost-toric base in the second row of Figure 1. The toric fan \(\Delta_{dP_3}\) associated to this moment polytope is generated by the vertices \((1, 0), (0, 1), (-1, 0), (0, -1), (-1, 1), (1, -1)\). The quiver \(Q_\Delta\) is depicted in Figure 26.

![Figure 26: Quiver associated to the fan of \(Bl_3(\mathbb{C}P^2)\).](image)

Let \(\mu_1, \mu_2, \mu_3\) be the ordered mutations at the vertices \(v_1, v_5\) and \(v_2\), so \(s = 3\) in this example. The toric fan \(\Delta_{dP_3}\) becomes a triangular fan after these three mutations. Indeed, the fan \(\mu_3(\mu_2(\mu_1(\Delta_{dP_3})))\) is that of \(\mathbb{P}(1, 2, 3)\). The partition of the vertex set \(Q_{\Delta_{dP_3}}\) is:

\[
T_1 = \{3, 5\}, \quad T_2 = \{1, 2, 4\}, \quad F = \{6\}.
\]

Let us now follow the algebraic recipe above. First, initialize the six cluster variables

\[
x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1.
\]
In what follows, we write the mutations for the cluster variables without specifying the mutation in the quiver, which is assumed to occur as well in each mutation. For instance, Figure 27 draws the quiver after mutating at the variables $x_1, x_5$ and $x_2$, in this order.

The first three mutations are at $x_1, x_5$ and $x_2$, with resulting cluster variables (still) being:

\[
\begin{align*}
&\begin{cases}
  x_1 = 1, & x_3 = 1 \\
  x_2 = 1, & x_5 = 1 \\
  x_4 = 1, & x_6 = 1
\end{cases}, \\
&\begin{cases}
  x_1 = 1, & x_3 = 1 \\
  x_2 = 1, & x_5 = 1 \\
  x_4 = 1, & x_6 = 1
\end{cases},
\end{align*}
\]

but the current seed is that associated to the quiver in Figure 27. In this precise seed, we start mutating according to the steps. First, we mutate at $x_1, x_2, x_4$, and we obtain:

\[
\begin{align*}
&\begin{cases}
  x_1 = 1, & x_3 = 1 \\
  x_2 = 1, & x_5 = 1 \\
  x_4 = 1, & x_6 = 1
\end{cases},
\end{align*}
\]

Then we mutate at the variables $x_3, x_5$, which leads to

\[
\begin{align*}
&\begin{cases}
  x_1 = 1, & x_3 = 2 \\
  x_2 = 1, & x_5 = 2 \\
  x_4 = 1, & x_6 = 1
\end{cases},
\end{align*}
\]

Mutating at $x_1, x_2, x_4$ sets the cluster variables to

\[
\begin{align*}
&\begin{cases}
  x_1 = 1, & x_3 = 2 \\
  x_2 = 3, & x_5 = 2 \\
  x_4 = 3, & x_6 = 1
\end{cases}.
\end{align*}
\]

The next few numerical seeds of these iterative process are:

\[
\begin{align*}
&\begin{cases}
  x_1 = 3, & x_3 = 7 \\
  x_2 = 3, & x_5 = 7 \\
  x_4 = 3, & x_6 = 1
\end{cases} \rightarrow \begin{cases}
  x_1 = 11, & x_3 = 7 \\
  x_2 = 11, & x_5 = 7 \\
  x_4 = 11, & x_6 = 1
\end{cases} \rightarrow \begin{cases}
  x_1 = 11, & x_3 = 26 \\
  x_2 = 11, & x_5 = 26 \\
  x_4 = 11, & x_6 = 1
\end{cases} \rightarrow \begin{cases}
  x_1 = 41, & x_3 = 26 \\
  x_2 = 41, & x_5 = 26 \\
  x_4 = 41, & x_6 = 1
\end{cases},
\end{align*}
\]
and the first few values of the sequence $a_n$ are:

\[ \{a_n\} = \{1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, 362, 571, 1351, 2131, 5042, 7953, 18817, 29681, \ldots \} \]

The cluster variable mutation is dictated by the transformation:

\[ x_{n+3} = \frac{x_{n+1}x_{n+2}+1}{x_n}, \]

and satisfies the linear recurrence $x_n = 4x_{n-2} - x_{n-4}$. This is Sequence A005246 in the Sloane notation.

Consider the Markov weighted sequence

\[ A_n = \begin{cases} A_n = 2a_n^2, & \text{n odd,} \\ A_n = 3a_n^2, & \text{n even.} \end{cases} \]

This is the sequence associated to the balanced quivers and exactly matches our arithmetic in Subsection 3.2. In particular, the sharp points of the infinite stairs associated to the toric diagram are given by the consecutive quotients for \( \{A_n\} \), which tend to $2 + \sqrt{3}$. 

\[ \square \]

References


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