This examination document contains 8 pages, including this cover page, and 7 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (15 points) Prove the following two statements.
   (a) (7 points) Show that for every $n \in \mathbb{N}$ with $n \geq 2$:
   \[
   n^2 \leq 4 \cdot \left( \frac{n}{2} \right)^2.
   \]
   (b) (8 points) Prove that there do no exist integers $x, y \in \mathbb{Z}$ such that
   \[
   15x^2 - 7y^2 = 301.
   \]
2. (15 points) Show that the following sequences \((x_n)\) converge to the indicated limit by using the \(\varepsilon\)-definition of the limit.

(a) (8 points) \(\lim_{n \to \infty} \frac{2^n + n!}{n^n} = 0\).

(b) (7 points) \(\lim_{n \to \infty} \frac{3n^7 + n + 4}{n^7 + 5n^2 + 9} = 3\).
3. (15 points) Let \((x_n), n \in \mathbb{N}\), be a sequence of real numbers that satisfies the recursion

\[ x_{n+1} = \frac{x_n + 3}{5}, \]

with the initial value \(x_1 = 1\).

(a) (8 points) Show that the sequence is decreasing and bounded below.

(b) (7 points) Show that the sequence is convergent and its limit is \(\frac{3}{4} \in \mathbb{R}\).
4. (15 points) Consider the set

\[ X = \{ q^4 + 2 : q \in \mathbb{Q} \cap (1, \sqrt{2}] \}. \]

(a) (8 points) Show that \( \inf(X) = 3 \) and \( \sup(X) = 6 \).

(b) (7 points) Give an explicit bijection \( f : \mathbb{Q} \cap (1, \sqrt{2}] \to X \).
(You must show it is a bijection.)
5. (15 points) Prove the following two statements.

(a) (8 points) Show that $\sqrt[3]{19}$ is not a rational number.

(b) (7 points) Find a sequence $(x_n)$ of irrational numbers such that $(x_n)$ is convergent and its limit is a rational number.
6. (15 points) Solve the following two problems:
   
   (a) (8 points) Consider the function
   \[ f : \mathbb{R}^+ \to (-\infty, 5], \quad f(x) = 5 - x^4, \]
   and show that it is an injection. Is it a surjection? (Prove your answer.)

   (b) (7 points) Give an injection \( f : \mathbb{Z} \to \mathbb{N} \) which is \textbf{not} a surjection.
7. (10 points) Prove or disprove the following assertions. Recall that the power set $P(X)$ of a set $X$ is the set of all the subsets of $X$.

(a) (5 points) The power set $P(\mathbb{R})$ of $\mathbb{R}$ is uncountable.

(b) (5 points) Let $X, Y$ be finite sets such that there exists an injection $P(X) \rightarrow P(Y)$ between their power sets. Then there exists an injection $g : X \rightarrow Y$. 