This examination document contains 9 pages, including this cover page, and 7 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

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1. (15 points) Prove the following two statements.
   (a) (7 points) Show that for every \( n \in \mathbb{N} \)
   
   \[ n^2 + 6n + 7 < 20n^2. \]

   **Solution.** The inequality is equivalent to \( 6n + 7 < 19n^2 \). Let us prove this latter inequality by induction. The base case is \( n = 1 \), which reads \( 6 + 7 < 19 \), and it is certainly true.

   For the induction step, we assume that the inequality \( 6n + 7 < 19n^2 \) holds, and we try to show \( 6(n + 1) + 7 < 19(n + 1)^2 \). Indeed,

   \[ 6(n+1)+7 < 19(n+1)^2 \iff 6(n+1)+7 < 19(n^2+2n+1) \iff 6n+13 < 19n^2+38n+19. \]

   The inequality on the rightmost equivalent \( 6n + 13 < 19n + 19 < 19n^2 + 38n + 19 \) holds because \( 6n + 13 < 19n + 19 < 19n^2 + 38n + 19 \). \( \square \)

   (b) (8 points) Prove that for all \( n \in \mathbb{N} \), \( 13^n - 6^n \) is divisible by 7.

   **Solution.** Let us work with arithmetic modulo 7. Then we have

   \[ 13^n - 6^n \equiv (-1)^n - (-1)^n \equiv 0 \pmod{7}, \]

   and thus the expression is always divisible by 7. \( \square \)
2. (15 points) Show that the following sequences \((x_n)\) converge to the indicated limit by using the \(\varepsilon\)-definition of the limit.

(a) (8 points) \[\lim_{n \to \infty} \frac{2n^2 + 1}{n^2 + 1} = 2.\]

**Solution.** For any \(\varepsilon > 0\), we want to show that

\[\left| \frac{2n^2 + 1}{n^2 + 1} - 2 \right| < \varepsilon, \quad \text{for } n \gg 1.\]

Let us rearrange this expression to

\[\left| \frac{2n^2 + 1}{n^2 + 1} - 2 \cdot \frac{n^2 + 1}{n^2 + 1} \right| < \varepsilon, \quad \text{for } n \gg 1,
\]

which reads

\[\left| \frac{-1}{n^2 + 1} \right| < \varepsilon, \quad \text{for } n \gg 1.\]

It thus suffices to show that

\[\frac{1}{n^2 + 1} < \varepsilon, \quad \text{for } n \gg 1.\]

Indeed, by Proposition 10.4, we can choose \(n \in \mathbb{N}\) large enough such that

\[\frac{1}{n} < \varepsilon, \quad \text{for } n \gg 1.\]

For that same \(n \in \mathbb{N}\) we will then have

\[\frac{1}{n^2 + 1} < \frac{1}{n} < \varepsilon, \quad \text{for } n \gg 1,
\]

as required. \(\square\)
(b) (7 points) \( \lim_{n \to \infty} \frac{6^n}{n!} = 0. \)

**Solution.** For any \( \varepsilon > 0 \), we want to show that

\[
\left| \frac{6^n}{n!} \right| < \varepsilon, \quad \text{for } n \gg 1.
\]

Given that \( \varepsilon > 0 \), by Proposition 10.4, we can choose \( n \in \mathbb{N} \) large enough such that

\[
\frac{1}{n} < \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \varepsilon}{6^6}, \quad \text{for } n \gg 1.
\]

For that same \( n \in \mathbb{N} \), we can use the inequality

\[
\frac{6^n}{n!} = \frac{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot \ldots \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \ldots \cdot (n-1) \cdot n} < \frac{6^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot n}
\]

where we are using \( \frac{6}{k} \leq 1 \) for all \( 6 \leq k \leq (n-1) \). Then we will have

\[
\frac{6^n}{n!} < \frac{6^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot n} < \varepsilon, \quad \text{for } n \gg 1,
\]

as we wanted to prove. \( \square \)
3. (15 points) Let \((x_n), n \in \mathbb{N}\), be a sequence of real numbers that satisfies the recursion

\[ x_{n+1} = \frac{x_n + 1}{4}, \]

with the initial value \(x_1 = 1\).

(a) (8 points) Show that the sequence is decreasing and bounded below.

**Solution.** First, the sequence is bounded below by 0, since all the terms are positive. Second, let us show that \(x_n\) is a decreasing sequence.

By induction, the base case is \(x_1 > x_2\), which reads \(1 > \frac{1+1}{4} = 0.5\). The induction step is proven as follows. We assume that \(x_{n+1} > x_n\) and we want to show that \(x_{n+2} > x_{n+1}\). Indeed, by the recursive definition

\[ x_{n+2} > x_{n+1} \iff \frac{x_{n+1} + 1}{4} > \frac{x_n + 1}{4} \iff x_{n+1} > x_n, \]

and the latter inequality is true by the induction hypothesis. \(\square\)

(b) (7 points) Show that the sequence is convergent and find its limit.

**Solution.** By Part (a), the hypothesis of the Monotone Convergence Theorem (Theorem 10.19 and Project 10.20) are satisfied, and thus we know that any decreasing sequence which is bounded below is convergent.

In order to find its limit we apply the recipe in class. The limit \(L\) of \((x_n)\) must satisfy the same recursion as the sequence \((x_n)\). This implies that \(L\) satisfies

\[ L = \frac{L + 1}{4}, \]

which yields the unique solution \(L = \frac{1}{3}\). In consequence, we have proven that

\[ \lim_{n \to \infty} x_n = \frac{1}{3}, \]

as it is required. \(\square\)
4. (15 points) Consider the set

\[ X = \{\sqrt{n+1} - \sqrt{n} : n \in \mathbb{N}\} \]

(a) (8 points) Show that \( \inf(X) = 0 \) and \( \sup(X) = \sqrt{2} - 1 \).

**Solution.** Since the sequence \( \sqrt{n+1} - \sqrt{n} \) is decreasing in \( n \in \mathbb{N} \), the value of \( \sqrt{n+1} - \sqrt{n} \) will be largest for \( n = 1 \). This value is \( \sqrt{2} - 1 \in X \), and thus \( \sup(X) = \sqrt{2} - 1 \).

Let us now prove \( \inf(X) = 0 \). By contradiction, suppose that the infimum is \( 0 < \inf(X) \), so that 0 is not the greatest lower bound. By definition, \( \inf(X) \) is a lower bound for \( X \), and we will reach a contradiction with that, which will prove that.

By Proposition 10.4, since \( 4 \inf(X)^2 \) is a positive quantity, we can find an \( n \in \mathbb{N} \) such that

\[ \frac{1}{n} < 4 \inf(X)^2, \]

which is equivalent to

\[ \frac{1}{2\sqrt{n}} < \inf(X). \]

For that \( n \in \mathbb{N} \) we will have

\[ \sqrt{n+1} - \sqrt{n} < \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} < \inf(X). \]

This proves the inequality \( \sqrt{n+1} - \sqrt{n} < \inf(X) \), and since \( \sqrt{n+1} - \sqrt{n} \in X \) we have reached the contradiction that \( \inf(X) \) is a lower bound for \( X \). In consequence, \( \inf(X) = 0 \), as we wanted to show.

(b) (7 points) Show that \( X \) admits a bijection to \( \mathbb{Q} \times \mathbb{Q} \).

**Solution.** Since \( \mathbb{Q} \) is countable, there exists a bijection \( f : \mathbb{Q} \rightarrow \mathbb{N} \). Then, we can construct the function

\[ (f, f) : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{N} \times \mathbb{N}, \quad q \mapsto (f(q), f(q)), \]

which is a bijection since each component is \( f : \mathbb{Q} \rightarrow \mathbb{N} \), which is a bijection. Now \( \mathbb{N} \times \mathbb{N} \) is countable, e.g. by Corollary 13.16, or also directly because it is a subset of \( \mathbb{Z} \times \mathbb{Z} \) which, by Theorem 13.14, is countable.
5. (15 points) Prove the following two statements.
   
   (a) (8 points) Show that $\sqrt[5]{7}$ is not a rational number.

   **Solution.** Let us prove this by contradiction. Suppose that $\sqrt[5]{7} \in \mathbb{Q}$ then there exist two integers $p, q \in \mathbb{Z}$ such that
   $$\sqrt[5]{7} = \frac{p}{q}, \quad \text{i.e.} \quad 7q^5 = p^5,$$
   and we assume, after simplifying common factors, that $gcd(p, q) = 1$, i.e. they have no common divisor. Now we look at the equality
   $$7q^5 = p^5,$$
   and study what it tells us about divisibility.
   The left hand side is divisible by 7, and thus 7 divides $p^5$, which is the right hand side. Since 7 divides $p^5$, then 7 must divide $p$, since 7 is prime. If 7 divides $p$, then $7^5$ divides $p^5$, which is the right hand side of the equality. Thus, $7^5$ divides the left hand side $7q^5$. Since $7^5$ divides $7q^5$, it must be that 7 (and even $7^4$) divides $q^5$, and in consequence $q$ itself. The conclusion is that 7 divides $q$ and 7 divides $p$. This is a contradiction with the initial assumption that $gcd(p, q) = 1$, since 7 is a common divisor. \(\Box\)

(b) (7 points) Find a sequence of rational numbers that converges to $\sqrt{2}$.

   **Solution.** We can use the recipe for recursive sequences. For instance, consider $(x_n)$ defined by
   $$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}, \quad x_1 = 2.$$  
   The sequence $(x_n)$ is decreasing and bounded below, and hence by the Monotone Convergence Theorem it is convergent. The limit $L$ satisfies the equation
   $$L = \frac{1}{2}L + \frac{1}{L} \iff L^2 = \frac{1}{2}L^2 + 1 \iff \frac{1}{2}L^2 = 1 \iff L = \pm\sqrt{2},$$
   which is why we choose the initial recursion as we did. Now, the sequence $(x_n)$ is of positive real numbers, and thus the limit $L$ must be non-negative. Thus $L = \sqrt{2}$ and we have proven $\lim_{n \to \infty} x_n = \sqrt{2}$. \(\Box\)
6. (15 points) Consider the function

\[ f : \mathbb{Z} \rightarrow \mathbb{Q}, \quad f(x) = \frac{3}{x - 0.5}. \]

(a) (8 points) Show that \( f \) is an injection but not a surjection.

**Solution.** First, let us prove that \( f \) is an injection. Indeed, \( f(x) = f(y) \) implies \( x = y \) as follows:

\[
\frac{3}{x - 0.5} = \frac{3}{y - 0.5} \iff 3(y - 0.5) = 3(x - 0.5) \iff x = y.
\]

Let us now prove that \( f \) is not a surjection. Indeed, take for instance \( 3 \in \mathbb{Q} \). We want to show that there exists no \( x \in \mathbb{Z} \) such that \( f(x) = 3 \). For that, set the equation \( f(x) = 3 \), which reads

\[
\frac{3}{x - 0.5} = 3 \iff \frac{1}{x - 0.5} = 1 \iff x = 1.5.
\]

Thus, if \( f(x) = 3 \) we must have \( x = 1.5 \), but \( 1.5 \notin \mathbb{Z} \), and \( f \) is not surjective. \( \Box \)

(b) (7 points) Show that there exists a function \( g : \mathbb{Z} \rightarrow \mathbb{Q} \) which is a surjection but not an injection.

**Solution.** Since \( \mathbb{Q} \) and \( \mathbb{Z} \) are countable, there exist bijections \( f : \mathbb{N} \rightarrow \mathbb{Q} \) and \( g : \mathbb{N} \rightarrow \mathbb{Z} \), and thus it suffices to give a function \( h : \mathbb{N} \rightarrow \mathbb{N} \) which is a surjection but not an injection. We can take, for instance, the function

\[ h : \mathbb{N} \rightarrow \mathbb{N}, \quad h(n) = \lfloor n/3 \rfloor, \]

where \( \lfloor m \rfloor \), for \( m \in \mathbb{Q} \), denote the first natural number \( \lfloor m \rfloor \in \mathbb{N} \) smaller or equal than \( m \), i.e. the integral part of \( m \). \( \Box \)
7. (10 points) Prove or disprove the following assertions.

(a) (5 points) The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^3 \) is a bijection.

**Solution.** This is true. First, let us first prove that it is an injection. Indeed, 
\( f(x) = f(y) \) implies that \( x^3 = y^3 \), which in turn implies \( x = y \). This proves the 
function is injective.

Second, let us prove surjectivity. Let \( y \in \mathbb{R} \) be a real number, and choose \( x = \sqrt[3]{y} \).
Then we will have \( f(\sqrt[3]{y}) = y \) and thus the function is surjective. \( \square \)

(b) (5 points) Any injection \( f : X \rightarrow Y \) between sets \( X,Y \) of the same cardinality is 
necessarily a bijection. (The cardinality might not be finite.)

**Solution.** This is false, it is only true for sets of finite cardinality. The injection 
\( f : \mathbb{N} \rightarrow \mathbb{Z}, f(n) = n \) is not surjective. \( \square \)