Symplectic Topology of Affine Hypersurfaces

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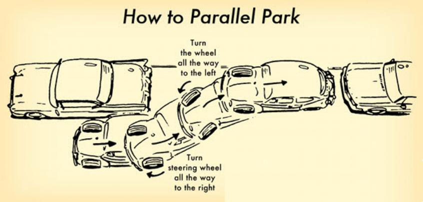
The importance of being contact

There exists a smooth path between two points in the plane $\mathbb{R}^2$.

Question: Can we also trace this path if we are skating or driving?



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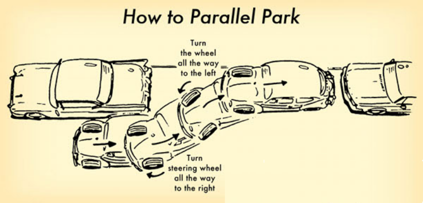

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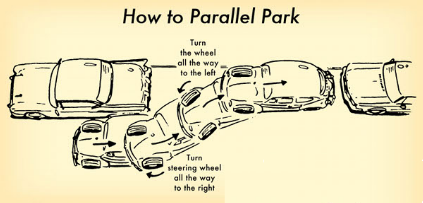

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Remark: Cats are also indebted to this phenomenon.

Legendrian knots

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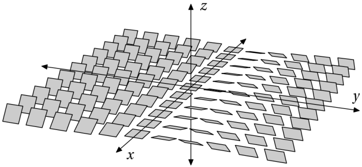

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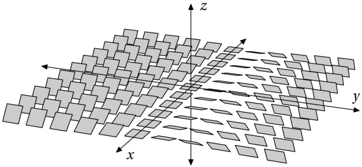

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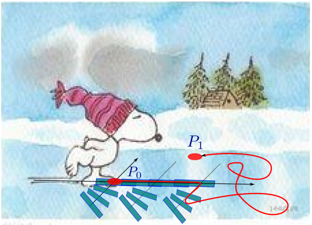
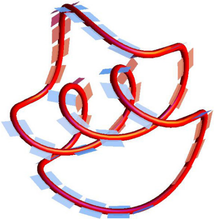
Embedded curves in $\mathbb{R}^3$ tangent to the plane field: **Legendrian knots**





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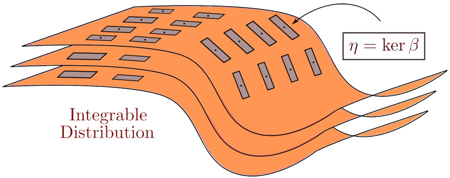

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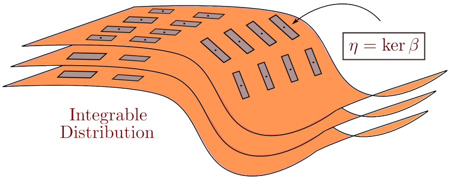

Figure : The distribution η is integrable.

- **No Integrable**: **no** integral surface exists, even locally.

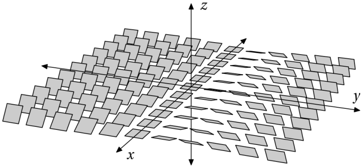

Figure : The distribution $\xi = \ker(dz - ydx)$ is **non–integrable**.

Integrability and algebra

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Theorem (Deahna 1849, Clebsch 1866, Frobenius 1877)

Let Y be a 3–fold and $\xi = \ker\alpha \subseteq TY$ a 2–plane distribution. Then the **rank** *of $d\alpha|_\xi$ measures* **integrability***:*

If $d\alpha = 0$ on ξ, then there exists a **surface** *S with $TS = \xi$.*

If $d\alpha \neq 0$ on ξ, the only possible submanifolds tangent to ξ are **knots***.*

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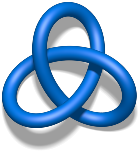
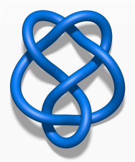
Now it is time to thank your multivariable calculus teachers!

A central advantage

Suppose we have a knot $\Lambda \subseteq \mathbb{R}^3(x, y, z)$, we need 3–dimensions:





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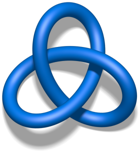

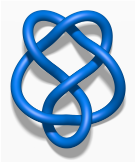

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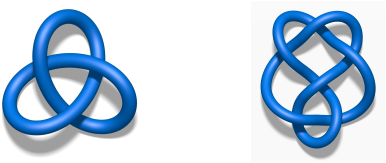

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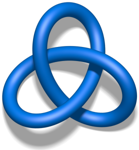
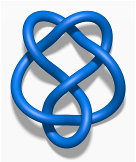

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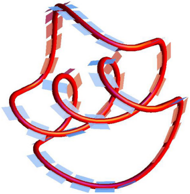
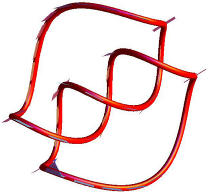

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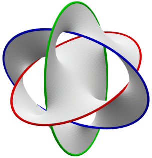

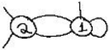

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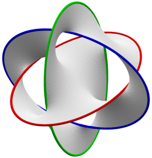

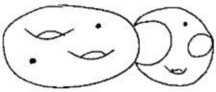

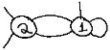

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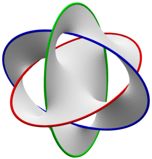

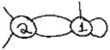

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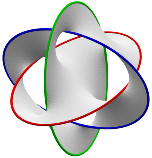

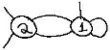

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- **Graph theory**: This is new! We have just discovered an invariant of a cubic graph, which enhances its chromatic polynomial to a DGA.

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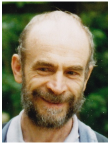


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 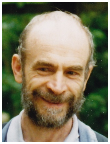 

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Since 1985, we have become very good at **distinguishing** contact structures using pseudoholomorphic invariants (e.g. A_∞-categories).

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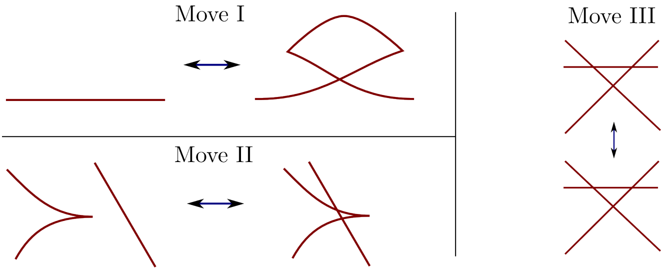

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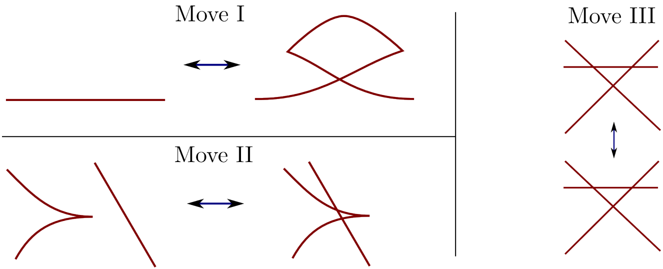

Now **pseudoholomorphic invariants** shall distinguish them!

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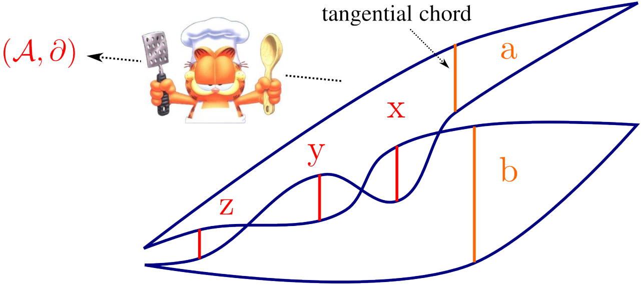

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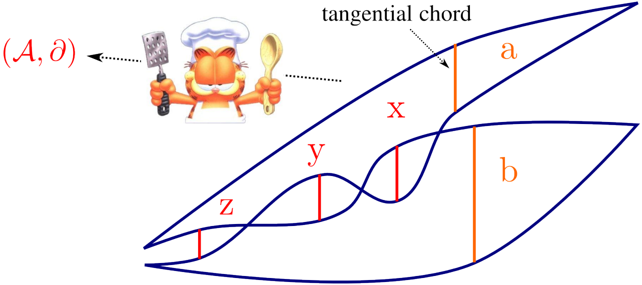

This constructs a **differential graded algebra** $(\mathcal{A}, \partial)$.

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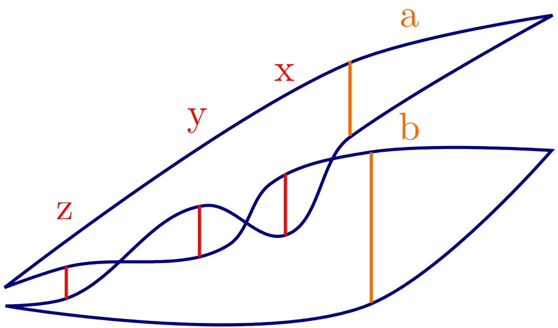

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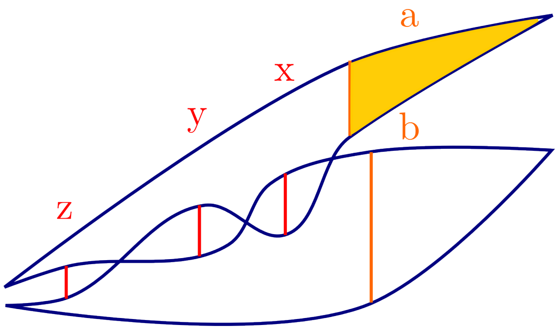

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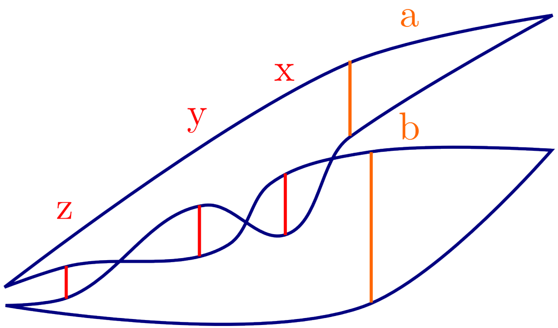

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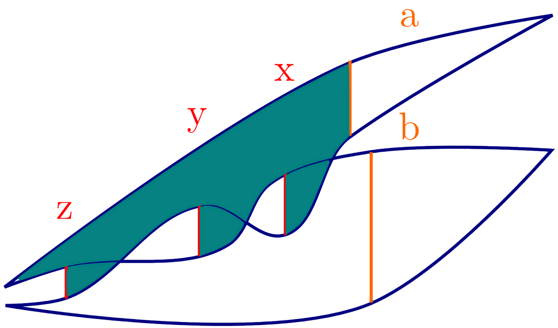

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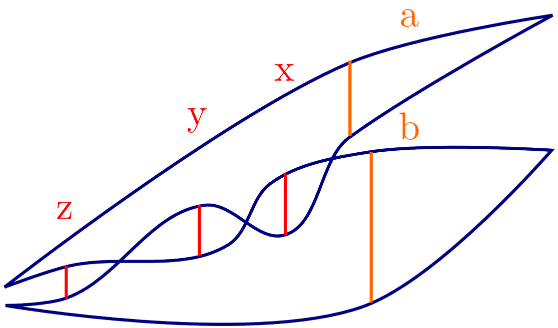

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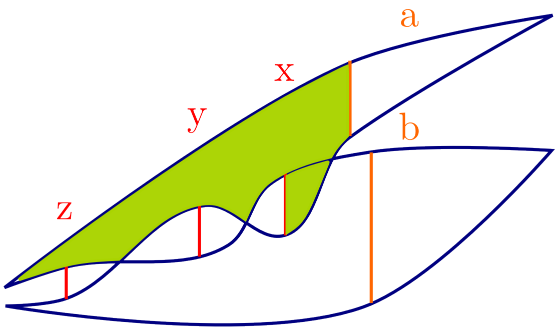

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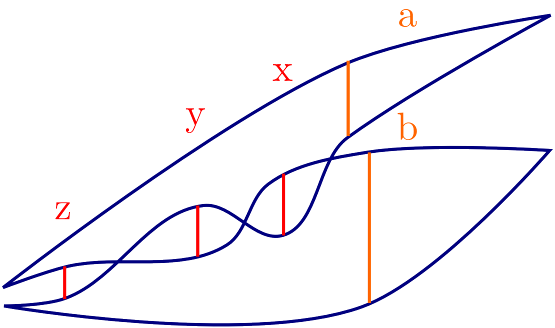

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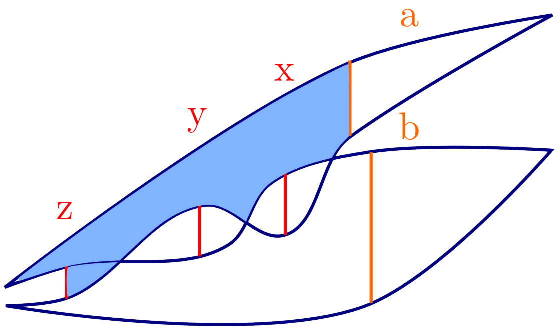

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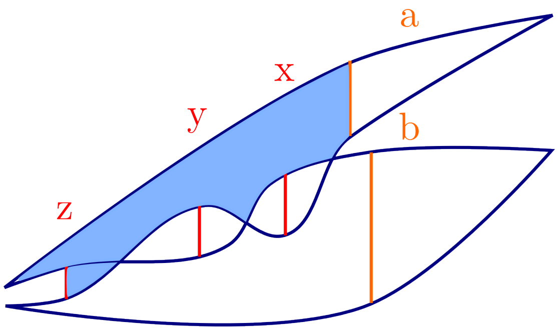

Homework: $\partial b = 1 + x + z + xyz$ i $\partial x = \partial y = \partial z = 0$.

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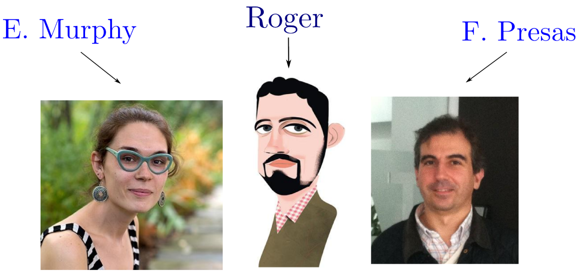

The **Zig–Zag** Team

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Theorem (M12,CMP14: The Zig–Zag Criterion)

Let (Y,ξ) be a contact structure and $\Lambda_1, \Lambda_2 \subseteq (Y,\xi)$ two Legendrians.

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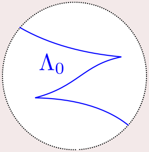

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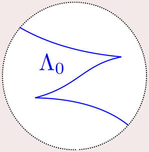

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Today's Tenet: If you can find a **zig–zag**, then you can **classify**.

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Basic guide to study **contact topology** and **Legendrian knots**

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Contact Topology →

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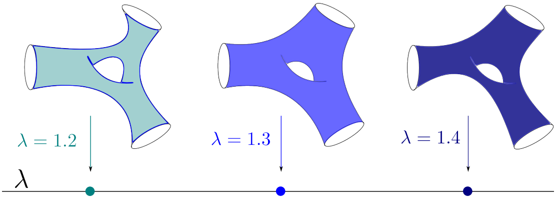

From Legendrian Knots to Complex manifolds

(CE 2012, CM 2016) The main correspondence

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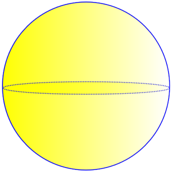

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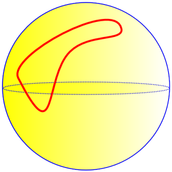

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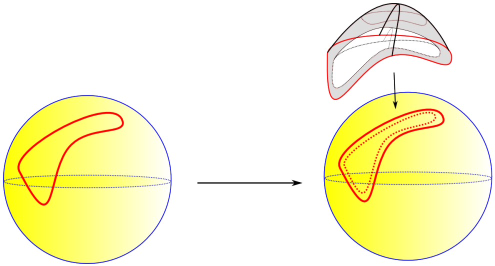

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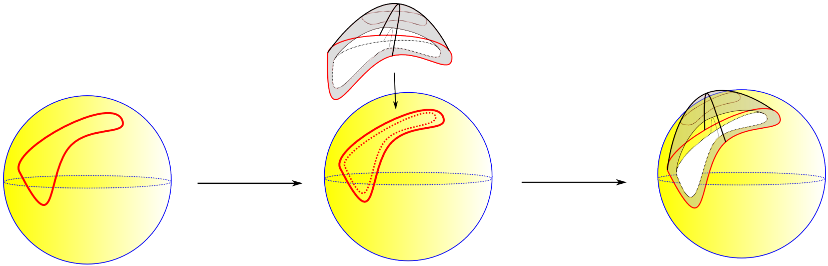

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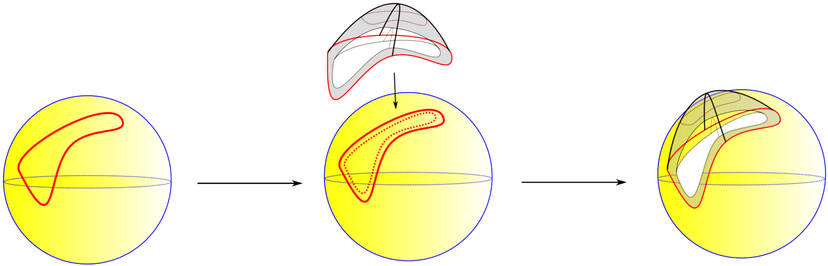

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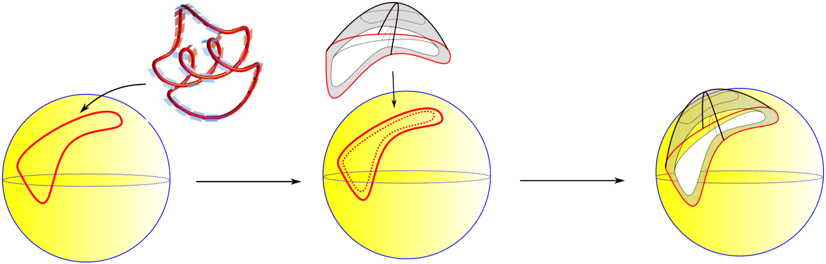

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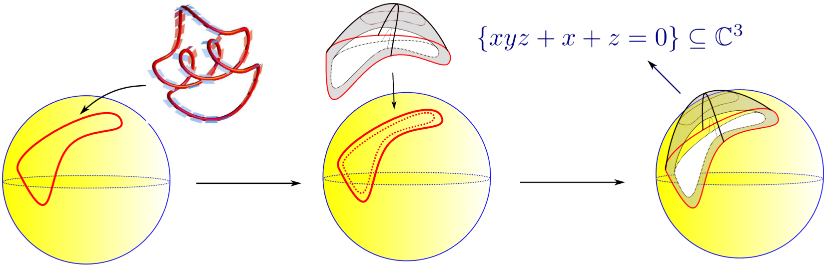

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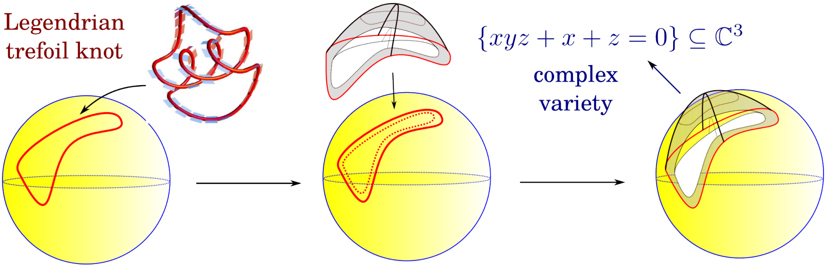

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From Complex Manifolds to Legendrian Knots

Theorem (CM 2016 – The Dictionary)

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Proof.

First choose a Lefschetz fibration $\pi : X \longrightarrow \mathbb{C}$.

1. Fix a basis of exact Lagrangian spheres $\{L_1, \ldots, L_r\}$ in the fiber F.
2. Choose a **second** Lefschetz fibration $\rho : F \longrightarrow \mathbb{C}$ and express these Lagrangian spheres $\{L_i\}$ as matching paths for the fibration ρ.
3. Given a **vanishing cycle** $V_i \subseteq (F, \lambda)$, draw the embedded path $\rho(V_i)$.
4. Now **plane combinatorics**: express each matching path $\rho(V_i)$ as a word in half–twists along the matchings paths of the $\{L_i\}$.
5. **Draw** the **front projection** of their Legendrian lifts. □

The Koras–Russell hypersurface

Theorem (CM 2016)

The **exotic** *Koras–Russell Cubic $\mathcal{C}$ deforms to $\mathbb{C}^3$.*

where here $\mathcal{C} = \{x + x^2y + z^2 + w^3 = 0\} \subseteq \mathbb{C}^4$.

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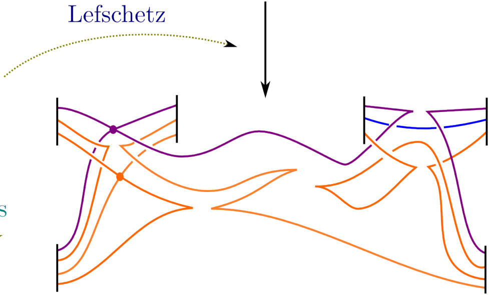

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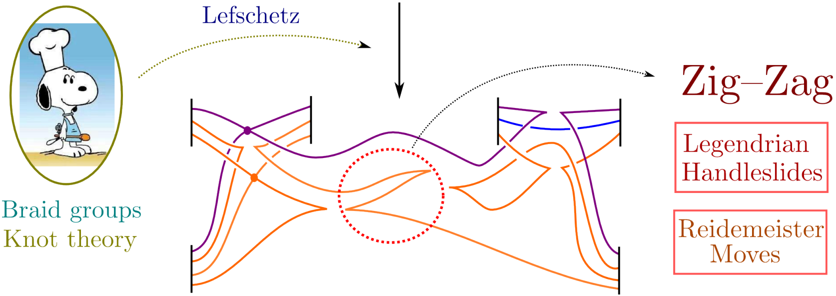

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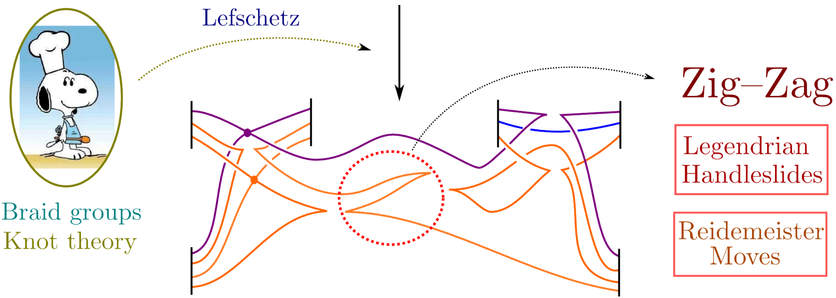

Hence, the Koras–Russell cubic is **deformation equivalent** to $\mathbb{C}^3$. □

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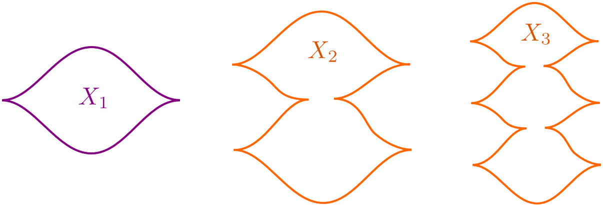

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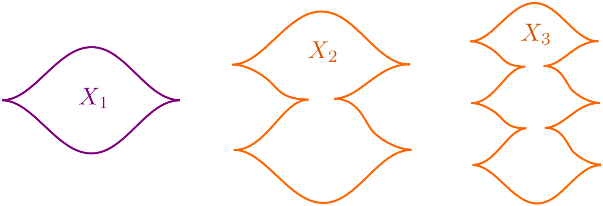

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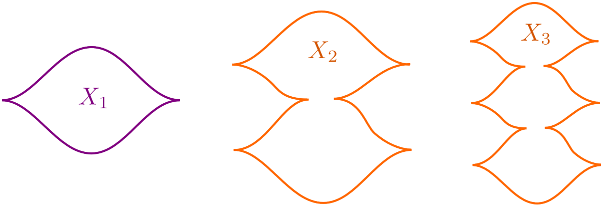

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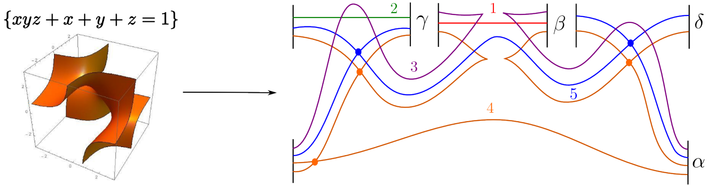

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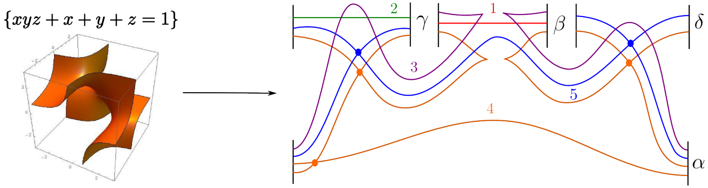

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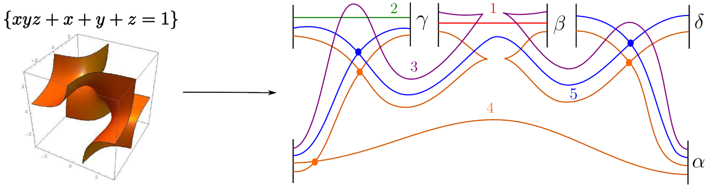

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Since the ring is commutative, the mirror is the algebraic manifold

$$X^* = \mathrm{Spec}(H^*(\mathcal{A}, \partial)) = \frac{\mathbb{C}[x, y, z]}{(xyz + x + y + z - 1)} = X \qquad \square$$

Epilogue

Contact topology and Legendrian knots

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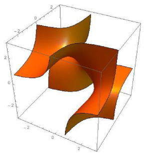

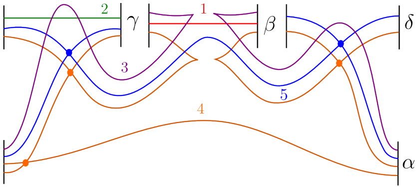

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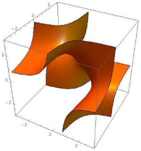

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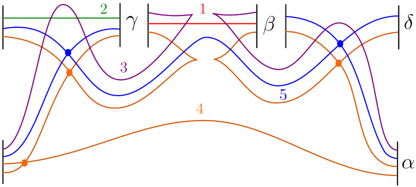

Then use $(\mathcal{A}, \partial)$ and **zig–zags** to prove results.

The end

Thanks a lot!