I. INTRODUCTION

Consider $a, b \in \mathbb{N}, a \leq b$. Let $E(a, b) \subseteq (\mathbb{R}^n, \omega_4)$ be the ellipsoid

$$E(a, b) := \left\{ (x,y) \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \leq 1 \right\}.$$

**GOAL:** Given a symplectic domain $(X, \omega)$, study symplectic embeddings $E(a, b) \hookrightarrow (X, \omega)$.

The INFINITE STAIRCASE PHENOMENON

Consider the function

$$c(x) = \inf \{ A_{E(1, a)} (X, \omega) \}$$

for $(X, \omega) = (B(1, a), \omega_4)$, $(B(1, 1), \omega_1)$ and $(\mathbb{R}^4)$. The function $c(x)$ is depicted in the graph.

Two remarks on $\text{Thm}$

1. The arithmetic of sharp points is controlled by a Diophantine equation, e.g., Minkowski $a^2 + b^2 + c^2 = 3abc$, for $(X, \omega) = (B(1, a), \omega_4)$.

2. See also "Infinite Staircases and Reflexive Polyhedra" by D. Cristina Goodman, Tara S. Holm, Alessia Mandini and A.R. Pires.

II. ALMOST TORIC GEOMETRY

Let $B \subseteq \mathbb{R}^n$ be a convex polytope (Fukaya-type $B$).

$$(X, \omega) \hookrightarrow (B, \omega_B).$$

Almost toric allows for flow singular fibres inside.

**Almost toric embeddings are also toric.**

SYMPLECTIC & LAGRANGIAN 2-Spheres

$S^2 \cong \mathbb{C}P^1$
The domains in our theorem:

- Above blue edges: symplectic $S^2$
- Above red edges: Lagrangian $S^2$

Then $(X,w)$ is the complement of these in the closed almost-toric manifold for the base.

### III. Sharp Ellipsoid Embeddings in $B^4$:

$(X,w) \cup B^4 \setminus \text{evolve}$

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(a) Mutations of $CP^2$ yield embeddings of $E(2,1; b^2) \to CP^2$ with $a^2 + b^2 + c^2 = 3abc (c=1)$

(b) Need to find line $L \subset CP^2$

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The general case: there are two situations that we encounter.

(i) Configurations of Lagrangian 2-spheres

- Use solution to Lagrangian isotopy problem for each configuration in $Bl, B3(P^2), P^2, Bl(14)$, and $Bl(15)$.

(ii) Configurations of symplectic 2-spheres

- Use solution to symplectic isotopy problem in these symplectic manifolds.

In both cases, need to build the right tropical curves: homology $\&$ int.

### Twisted Operations:

Input an almost-toric $f: (X,w) \to B$ and output a different almost-toric $g : (X,w) \to B$ for SANE $(X,w)$.

(1) $\sigma L(1,2)$ transformation on only one side.

(2) So only becomes a segment.

**Note:** we are allowed to split. If not smooth, still possible to smooth, up to floating sign.

Theorem (V.) Tropical diagrams in almost-toric base lift to configurations of symplectic curves as required.

In our $CP^2$ case, these tropical diagrams give linear $P^4$:

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**Sharp Ellipsoid Embeddings:**

- Case of $D^4 \times D^2$ and $E(1,2)$

1. Find tropical diagram $H_y$.
2. Argue uniqueness of equifying symplectic configuration.

In both cases, the sequence of mutated polygons is given by $e_i t^i x_i t^i x_i$. Note symmetry.
IV. TROPICAL DIAGRAMS: Two local models

1st: G. Khovanskii's balanced trivalent vertex

2nd: (new) Kreisel's model fibers (with multiplicity)

Example:

\[
(S^3, \xi_{ct}) \cong (L(1,1), \xi_{ct})
\]

For 2nd local model, in general, we have a lens space.

Two options:

(A) direct construction via local models, expressed as domains

(B) symplectic fibering of transverse link

WHAT'S NEXT?

(a) Sharp embedding sequence for neighborhoods of zero section in \( T^{*}S^2 \). Use \( \sim \) instead of a smooth vertex. (+ other domains?)

(b) Higher dimensional domains? Combinatorially, Atkin - Gates - Galkin - Kaprov\( \tilde{\chi} \) defines mutation.