Sharp Ellipsoid Embeddings & Toric Mutations

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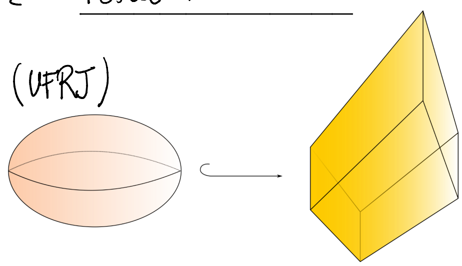
(arXiv 2004.13232)



I. INTRODUCTION

Consider $a, b \in \mathbb{N}$, $a \le b$.

Let $E(a,b) \subseteq (\mathbb{R}^4, \omega_{st})$ be the ellipsoid

$$E(a,b) := \{(x_1,y_1,x_2,y_2) \in \mathbb{R}^4 : \frac{x_1^2+y_1^2}{a} + \frac{x_2^2+y_2^2}{b} \le 1\}.$$

GOAL: Given a symplectic domain (X^4,ω), study symplectic embeddings $E(a,b) \hookrightarrow (X^4,\omega)$.

Known Results (selection)

(i) M. Gromov's Non-Squeezing (1985)

$\boxed{E(1,b) \overset{\text{sharp}}{\hookrightarrow} \mathbb{D}^2(A)\times\mathbb{R}^2 \text{ if and only if } 1 \le A.}$


D. McDuff


F. Schlenk

(ii) Constructions: D. McDuff ('08), L. Guth ('08), P. Biran ('97), O. Buse & R. Hind ('11, '13), F. Schlenk, D. Frenkel, D. Müller, and more!

(iii) Obstructions: Symplectic capacities, by M. Gromov, I. Ekeland, H. Hofer, C. Viterbo, E. Zehnder + more, M. Hutchings (ECH capacities)

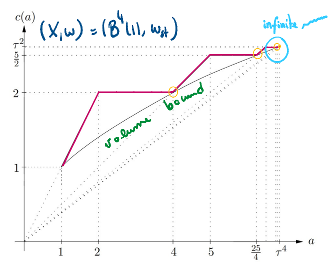
$\Rightarrow$ $\boxed{\text{infinite staircases}}$ for $(X,\omega) = (\mathbb{B}^4(1), \omega_{st})$, $(\mathbb{D}^2(1)\times\mathbb{D}^2(1), \omega_{st})$ and $E(2,3)$.

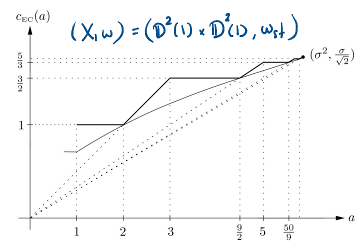
The INFINITE STAIRCASE PHENOMENON

(or how Fibonacci made an unexpected comeback)

Consider the function

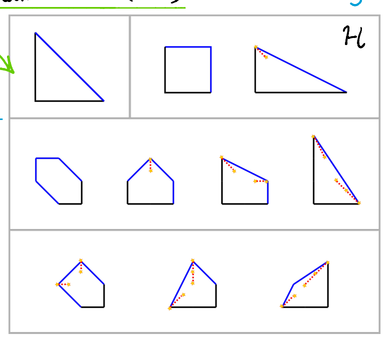
$$c(a) := \inf\{ A \in \mathbb{R} : E(1,a) \hookrightarrow (X, A\cdot\omega)\}.$$





THIS TALK: SHARP VOLUME EMBEDDINGS

Thm (C.-Vianna): The symplectic domains $(X,\omega) \in \mathcal{H}$ exhibit an infinite staircase of sharp ellipsoid embeddings.



Two ingredients:

$\boxed{\text{(1) Toric Mutations}}$

$\boxed{\text{(2) Symplectic Tropical Curves in almost-toric fibrations of } (X,\omega)}$

Two remarks on Thm

(1) The arithmetic of sharp points is controlled by a Diophantine equation, e.g. Markov eq^n

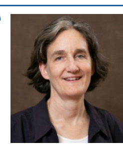
$$\alpha^2+\beta^2+\gamma^2 = 3\alpha\beta\gamma \quad \text{for } (X,\omega) = (\mathbb{B}^4(1), \omega_{st}).$$

(2) Please see also "Infinite Staircases and Reflexive Polygons" by D. Cristofaro-Gardiner, Tara S. Holm, Alessia Mandini and A.R. Pires

these cool people

II. ALMOST TORIC GEOMETRY

"Four dimensions from two in symplectic topology" by

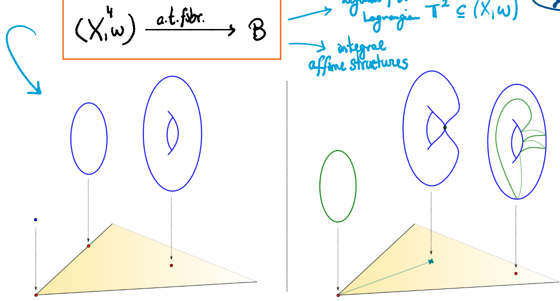

M. Symington

Let $B \subseteq \mathbb{R}^2$ be a convex polytope (Delzant type)

$\boxed{(X^4,\omega) \xrightarrow{\text{a.t.fibr.}} B}$

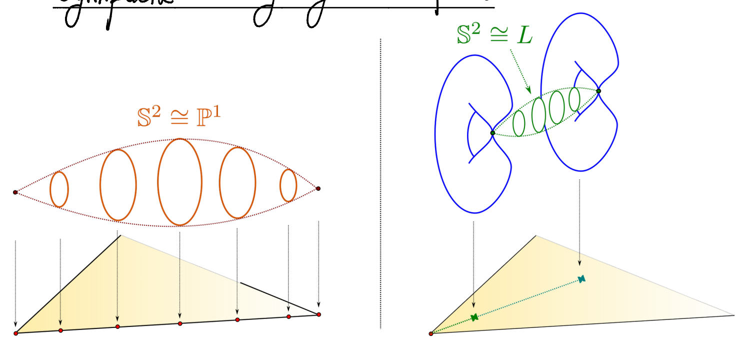
→ regular fiber Lagrangian $T^2 \subseteq (X,\omega)$

→ integral affine structures

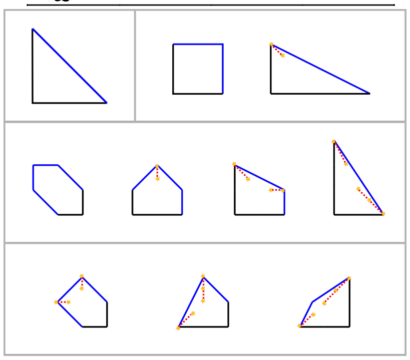


ALMOST-TORIC allows focus-focus singular fibers inside

Symplectic & Lagrangian 2-spheres



The domains in our theorem



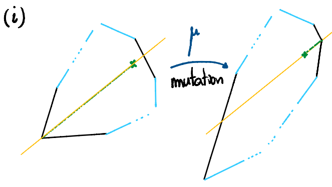
- Above blue edges symplectic S^2
- Above red edges Lagrangian S^2

Then (X,ω) is the complement of these in the closed almost toric manifold for the base.

Two Operations:

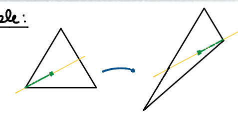
input an almost-toric $f:(X,\omega)\longrightarrow B$ and output a different almost-toric $g:(X,\omega)\longrightarrow B'$ $\boxed{\text{for SAME } (X,\omega)}$

(i)

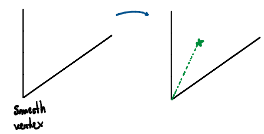


"$GL(2,\mathbb{Z})$" transformation on only one side. (So vertex becomes a segment.)

Example:

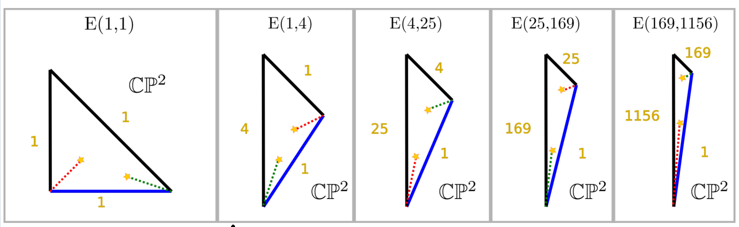


(ii)



Rmk: we are allowed to slide. If vertex not smooth, still possible to smooth, up to residual sing.

III. SHARP ELLIPSOID EMBEDDINGS in $B^4(1)$: $(X,\omega)=(B^4,\omega_{st})=\mathbb{CP}^2\setminus\{\text{line}\}$

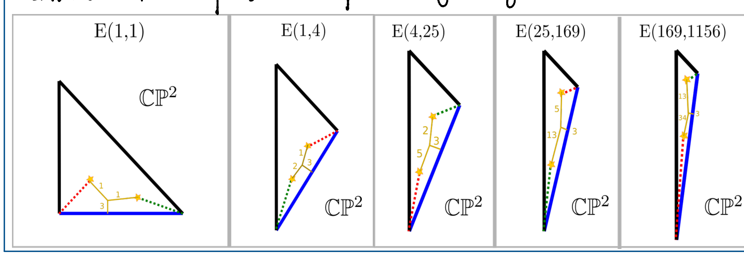


(a) Mutations of $\mathbb{CP}^2$ yield embeddings of $E(a^2,b^2)\hookrightarrow\mathbb{CP}^2$ with $a^2+b^2+c^2=3abc$ $(c=1)$

(b) NEED TO FIND LINE $L\subseteq\mathbb{CP}^2$

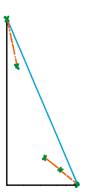
Thm (C.-Vianna): Tropical diagrams in almost-toric base lift to configurations of symplectic curves as required.

In our $\mathbb{CP}^2$ example, these tropical diagrams give linear $\mathbb{P}^1$:



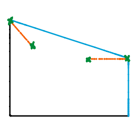
The general case: there are two situations that we encounter.

(i) Configurations of Lagrangian 2-spheres



* use solution to Lagr. isotopy problem for certain classes in $Bl_3(\mathbb{P}^2)$, $\mathbb{P}^2$, $Bl_1(\mathbb{P}^2)$ and $\mathbb{P}^1\times\mathbb{P}^1$.

(ii) Configurations of symplectic 2-spheres

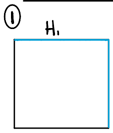


* use solution to symplectic isotopy problem in these symp. manifolds.

In both cases, need to build the right tropical curve: homology & int.

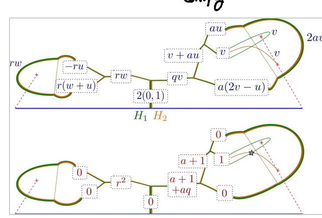
SHARP ELLIPSOID EMBEDDINGS: the cases of $\mathbb{D}^2\times\mathbb{D}^2$ and $E(1,2)$

1

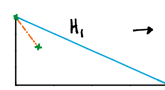


H_1 → Find tropical diagram

H_2 → Argue uniqueness of equisingular symplectic configuration.



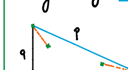
2



→ Find tropical diagram

→ Uniqueness for both symplectic and Lagrangian S^2.

In both cases the sequence of mutated polygons is given by $1+q^2+2r^2=4qr$, $p=1$.



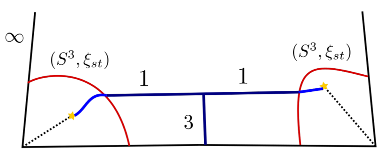
Frenkel-Müller's PELL STAIRCASE

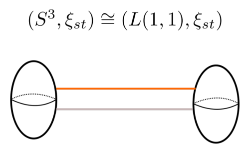
IV. TROPICAL DIAGRAMS : two local models

1st : G. Mikhalkin's balanced trivalent vertex

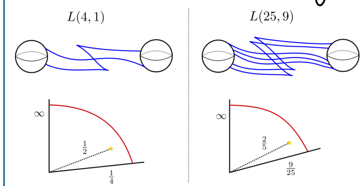
2nd : (new) Arrival at nodal fibers (with multiplicity).

Example :





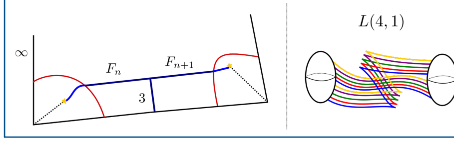
For 2nd local model, in general, we have a lens space.



Two options :

(A) direct construction via local models, expressed as dimers

(B) symplectic fillings of transverse link



WHAT'S NEXT ?

(a) Sharp embedding sequences for neighborhoods of zero section in T^*S^2. Use  instead of a smooth vertex. (+ other domains ?)

(b) Higher dimensional staircases ? Combinatorially, Akhtar - Coates - Galkin - Kasprzyk define mutation.

THE END

Thank you !