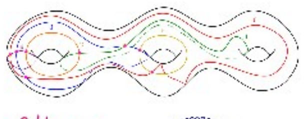
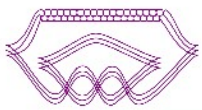
Legendrian Knots & Lagrangian fillings





R. Casals (UC Davis) @ Northwestern University G&P Seminar

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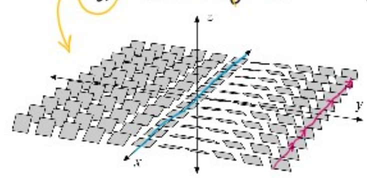
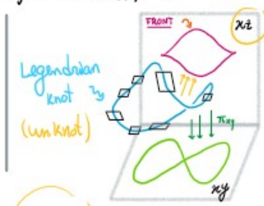
§1. INTRODUCTION : LEGENDRIAN KNOTS in $\mathbb{R}^3$

$\xi_{st} := TS^3 \cap i\,TS^3$

Def: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field $\xi_{st} := \mathrm{Ker}\{dz - y\,dx\}$. It compactifies to $(\mathbb{S}^3, \xi_{st})$. ■





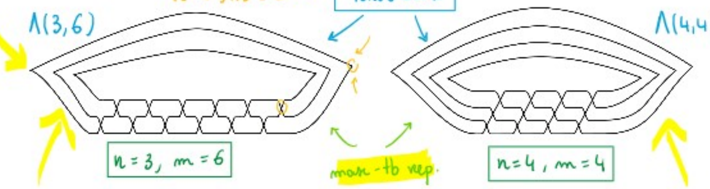
Def: A knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{LEGENDRIAN}}$ if $TK \subseteq \xi_{st}$. ■

Front projections: since $TK \subseteq \mathrm{Ker}\{dz - y\,dx\}$, we have $\boxed{y = \partial_x z(x)}$ on K.

⇒ the projection $\pi_{xz}(K)$ recovers K up to Legendrian isotopy.

$\pi_{xz}(K)$ ‼ the front of K

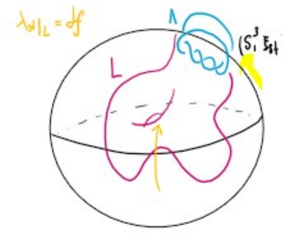
Two Examples : smoothly $\{z^n = w^m\} \cap S^3 \subseteq \mathbb{C}^2$ ⟵ $\boxed{\text{LEGENDRIAN TORUS LINKS}}$: $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$



LAGRANGIAN FILLINGS : let $\Lambda \subseteq (S^3, \xi_{st})$ be a Leg. link.

Def: A Lagrangian filling $L \subseteq (\mathbb{D}^4, \omega_{st})$ is an embedded exact Lagrangian surface in $\mathbb{D}^4$ with boundary $\partial L = \Lambda$ in $\partial\mathbb{D}^4 = S^3$. ■

$\lambda_{st}|_L = df$



Salient Facts :

(1) A Λ might or might not have a Lagr. filling.

(2) If ∃L filling Λ then $\boxed{g(L) = g_s(\Lambda)}$. different than smooth top.!

(3) (Eliashberg - Polterovich 1996) let $\Lambda = \Lambda_0$ be the max-tb standard unknot. Then ∃! L filling (the flat disk) up to Hamiltonian isotopy.

(4) Lagr. fillings are the objects of $\mathcal{W}(\mathbb{C}^2, \Lambda)$, the wrapped Fukaya category stopped at Λ. (See also Sh_Λ.)

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§2. 2020 Developments : the discovery of INFINITELY MANY LAGR. FILLINGS

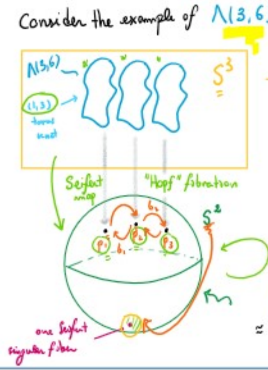
Thm. (C.-Gao 20) The Legendrian torus links $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$ have ($n \geq 3$, $m \geq 6$ or (4,4), (4,5)) infinitely many distinct Lagrangian fillings. [$\Lambda(3,6)$ has a $PSL_2(\mathbb{Z})$ worth and $\Lambda(4,4)$ a $M_{0,4}$]
In fact, ∃ ∞'ly many hyperbolic and satellites knots with this property !

Cor : There exists an abundance of Stein 4-manifolds W^4 homotopic to the 2-sphere $\mathbb{S}^2$ with ∞'ly many Lagrangian surfaces of genus g (and no Lagr. surface of genus < g). ■

Rmk : the Thm. is stronger than this, constructing a $PSL_2(\mathbb{Z})$ subgroup of the Lagrangian concordance monoid (acting faithfully on Lagr. fillings). Also, all the filling will be explicit !

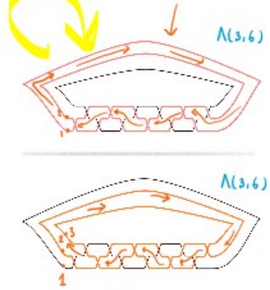
Torus links $\Lambda(n,m)$ & LEGENDRIAN Loops

Consider the example of $\Lambda(3,6)$, as follows :



$\boxed{\sigma_1 \text{ acts as :}}$ → $PSL_2\mathbb{Z}$ ← $\boxed{\sigma_2 \text{ acts as :}}$

action of mapping class group $MCG(\mathbb{R}^2, p_1, p_2, p_3) \cong$
$\cong Br_3/Z \cong PSL_2(\mathbb{Z}) = \langle \sigma_1, \sigma_2 \rangle$ ⇝ other interesting Br gp.
$\mathbb{Z}_2 * \mathbb{Z}_3$



The candidate Lagrangian fillings :

* why am I doing $\Lambda(3,6)$ and not smaller ?

Step 1 : Consider the graph $gr(\sigma_1)$ of the Legendrian loop associated to σ_1.

Step 2 : Concatenate $gr(\sigma_1)$ as many times as needed on top of a fixed Lagrangian filling L.

⟹

(1) L is smoothly isotopic to $L \cup gr(\sigma_1)$

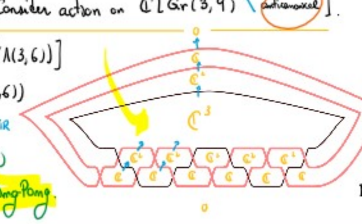
(2) Are the Lagrangian fillings $L \cup gr(\sigma_1)^{\ell} \neq L \cup gr(\sigma_1)^{k}$? (if $\ell \neq k$)

THE SCHEME OF PROOF OF THE THEOREM : I will sketch it for $\Lambda(3,6)$.

STEP I : Since σ_1 acts as an isotopy $\varphi^t_{\sigma_1}$, it acts on any invariant associated to $\Lambda(3,6)$. In particular, it gives a self-equiv. on Sh_Λ.

STEP II : The moduli $\mathcal{M}^{fr}(\Lambda(3,6))$ of framed sheaves embeds as the open positroid cell in $Gr(3,9)$. Consider action on $\mathbb{C}[Gr(3,9) \setminus \text{anticanonical}]$.

STEP III : Show that ∃ Plücker fct $\in \mathbb{C}[\mathcal{M}^{fr}(\Lambda(3,6))]$ such that the group $\langle \sigma_1, \sigma_2 \rangle \in \pi_1(\mathrm{Lag}(3,6))$ acts faithfully in the orbit. THIS REQUIRES A FAIR AMOUNT OF UNDERSTANDING OF σ_1, σ_2 & $Gr(3,9)$ → for $Gr(3,9)$ we use G. Kuperberg's SL_3-web basis & Ping-Pong.



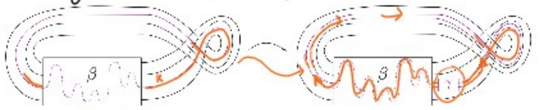
§3. Two more discoveries: ϑ-loops & Weave methods (+DT by H. Gao & collab.)

with L. Ng (Fall '20) — with E. Zaslow (Spring '20)

∞'ly many fillings I: ϑ-loops ⟵ generalizes the loops constructed above to many more links

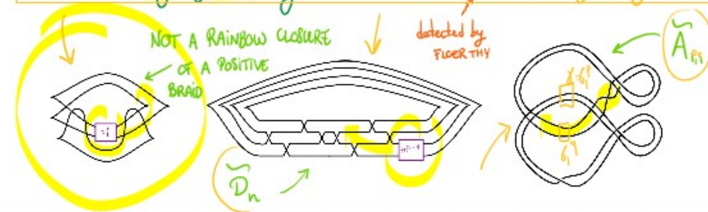
Let Λ be a Leg. link and $K \subseteq \Lambda$ a component.

E.g.



Then satellite $\gamma = \sigma_1^n$ along K. This creates a Legendrian loop by moving the purple box with γ inside around. ⟶ we call them ϑ-loops.

Thm: (C.-Ng in progress) Let Λ be any of the Legendrians links below. (among many!)
Then the ϑ-loops associated to the purple box monodromy have ∞ order when acting on the augmentation $\varepsilon_L : \mathcal{A}_\Lambda \longrightarrow \mathbb{Z}[H_1(L,\mathbb{Z})]$ associated to a certain Lagrangian filling of Λ. (In particular, ∞'ly many fillings!)



∞'ly many fillings II: Lagrangian skeleta & Weaves

A geometrization of cluster algebras: a Lagr. skeleton for $(\mathbb{D}^4, \Lambda)$ — Weinstein pair

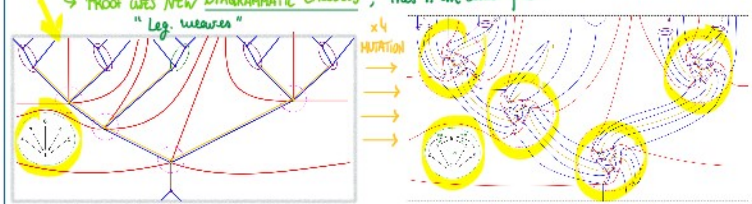
STRATEGY: Suppose $L \subseteq (\mathbb{D}^4, \omega_{st})$ is a Lagrangian filling and $\exists$ Lagr. 2-disks $\mathbb{D}^2 \subseteq \mathbb{D}^4 \setminus L$ with boundary in L. Then we can SURGER THE $\mathbb{D}^2$'s.

(a) Thm. (C'20): This skeleton exists if Λ is a max-tb algebraic link.

(b) Still, iterating surgeries typically creates immersed curves ⟶ hard to find $\mathbb{D}^2$'s to surger!

Thm: (C.-Zaslow Spring '20) There exist Legendrian links $\Lambda \subseteq (\mathbb{S}^3, \xi_{st})$ for which the surgery strategy can be implemented. In particular, with a quiver Q_Λ of infinite mutation type ($\Longrightarrow$ ∞'ly many Lagr. fillings)

↳ PROOF USES NEW DIAGRAMMATIC CALCULUS, this is an example:
"Leg. weaves"



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§4. ADE Conjecture (see also BCFG cases!) ⟵ arXiv 2009.06737

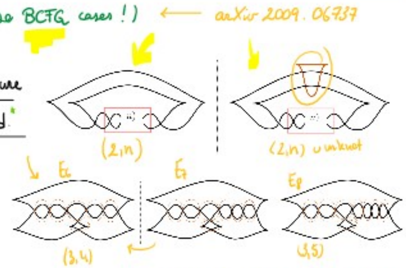
Conj. (2020) Let $\Lambda \subseteq (\mathbb{S}^3, \xi_{st})$ be a closure of a positive braid and Q_Λ connected.

Then

A (i) $\Lambda \cong \Lambda(A_n)$ and Λ has precisely → $\frac{1}{n+2} \cdot \binom{2n+2}{n+1}$ Lagr. fill.

D (ii) $\Lambda \cong \Lambda(D_n)$ and Λ has precisely ?→ $\frac{3n-2}{n} \cdot \binom{2n-2}{n-1}$ Lagr. fill.

E (iii) $\Lambda \cong \Lambda(E_6), \Lambda(E_7)$ or $\Lambda(E_8)$ with ↪ 833, 4160 and 25080 Lagr. fillings.



OR

(iv) Λ has ∞'ly many Lagr. fillings.

[GSW, GSW2]

SUMMARY OF TECHNIQUES

they all prove ∞'ly Lagr. fill.

Direct methods: (C.-Gao '20)
- Yield the strongest results (e.g. $PSL_2\mathbb{Z}$)
- they are hard to prove ⇝ categorical faithfulness? (see C. Fraser & B. Keller)

Lagr. skel & weaves: (C.-Zaslow Spring '20)
- only using the mutation class of Q_Λ
- the diagrams can get seriously complicated
  - rel^n to Soergel calc. & Charact. varieties
  - work-in-progress by J. Hughes for weaves in the D_n-case

Floer theory: (C.-Ng Fall '20)
- the only method that can tackle links which are not positive.
- required developing the theory / $\mathbb{Z}$ ⟶ work-in-progress by S. Rubin for a char 2 argument

Cluster algebras: (Gao-Shen-Weng Fall '20)
- immediate if the DT transformation has ∞ order
- the case of non-positive links is a mystery. (currently only $\mathbb{Z}_2$) ⟶ Augmentation stacks are cluster? Hol. Sympl.?

THE END

Thank you!