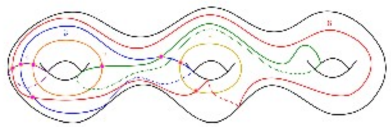
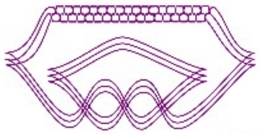
Lagrangian Skeleta & Isolated Plane Curve Singularities





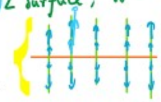
R. Casals (UC Davis) @
Prof. Gusein-Zade's Seminar in Moscow

§1. Symplectic 4-manifolds & Lagrangian Skeleta

Let (X^4,ω) be a symplectic 4-manifold: ω closed 2-form in X with $\omega^2\neq 0$.

Weinstein type: open symplectic (W,λ) with $\omega=d\lambda$, and $\varphi: W \xrightarrow{\text{Morse}} \mathbb{R}$.
Geometrically, X_λ "Liouville" v.f. and φ s.t. X_λ is pseudogradient.

Example: (1) Σ surface, $W=T^*\Sigma$.



(2) Any smooth affine surface: $W^4=\{xyz+x+z=1\}\subseteq\mathbb{C}^3$

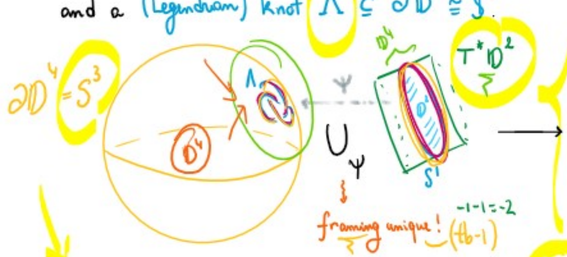
Closed (X,ω): Any projective surface with $\omega=\omega_{FS}$, e.g. $(X,\omega)=(\mathbb{CP}^2,\omega_{FS})$ or $(\mathbb{P}^1\times\mathbb{P}^1,\omega_{FS}\oplus\omega_{FS})$

Fact: (Donaldson, Giroux) Given any (X^4,ω) closed $\exists\, C^2\subseteq X^4$ closed symplectic surface s.t.
$$X=\underbrace{(X\setminus C)}_{\text{Weinstein}}\cup \underbrace{C}_{\text{closed symp.}}$$ ∎

Ex: • $\mathbb{P}^2\setminus \text{line}\cong\mathbb{C}^2$, • $(\mathbb{P}^1\times\mathbb{P}^1)\setminus\text{diagonal}\cong T^*S^2$
• $\mathbb{P}^2\setminus\text{conic}\cong T^*\mathbb{RP}^2$, • $(\mathbb{P}^1\times\mathbb{P}^1)\setminus\{pt\}\times\mathbb{P}^1$ not Weinst.

An important class of Weinstein 4-manifolds:

Consider the symplectic 4-ball $(\mathbb{D}^4,\omega_{st})\cong(T^*\mathbb{D}^2,d\lambda_{st})$ "$\cong$" $(\mathbb{C}^2,\omega_{st})$, and a (Legendrian) knot $\Lambda\subseteq\partial\mathbb{D}^4\cong S^3$.



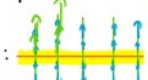
Example: Choose Λ the max-tb Legendrian unknot $\Lambda=\Lambda_0\subseteq S^3$, then
$W(\Lambda_0)\cong(T^*S^2,\omega_{st})\cong\{x^2+y^2+z^2=1\}\subseteq\mathbb{C}^3$. (Exercise: $W(\Lambda_{Hopf})=\{x^3+y^2+z^2=1\}$) plumbing of 2 T^*S^2

Lagrangian skeleta

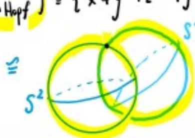
Not every W^4 is $W^4\cong T^*\Sigma$, Σ smooth surface. But!

Given the Liouville v.f. X_λ, we can flow backwards to retract W^4 to a (typically) singular surface. Since X_λ expands ω_{st}, a Lagrangian surface $\mathbb{L}$. "the skeleton"

Example: (i) If $W=T^*\Sigma$ and $\lambda=\lambda_{st}$, then $\mathbb{L}=\Sigma$:



(ii) If $W=W(\Lambda_{Hopf})=\{x^3+y^2+z^2=1\}\subseteq\mathbb{C}^3$ then $\mathbb{L}\cong$



= union of zero sections

(iii) If $W=\{xyz+x+z=1\}$, $\mathbb{L}\cong T^2$

M. Kontsevich's perspective, now more formalized and developed:

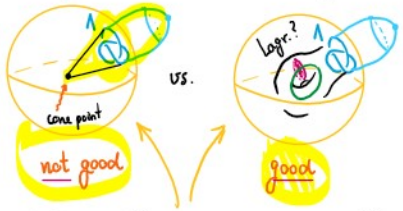
(1) $\mathbb{L}$ recovers W

(2) Many invariants of W can be computed "combinatorially" from $\mathbb{L}$.

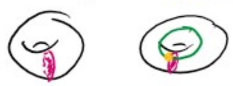
Arboreal Skeleta for $W = W(\Lambda_f)$: given W, many singular skeleta available.

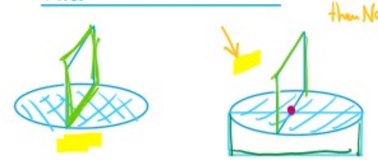
Only certain $\mathbb{L}$ allow for computations : good $\rightsquigarrow$ ARBOREAL SINGULARITIES (Turaev, then Nadler)

Example : Consider $W = W(\Lambda_{trefoil})$



In fact, $W(\Lambda_{trefoil}) \cong \{xyz + x + z = 1\}$.

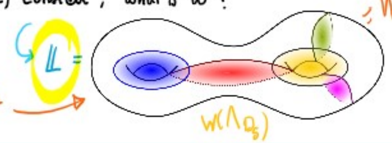




Example : (1) Smooth surface $\mathbb{L} = \Sigma$ gives cotangent bundle $W = T^*\Sigma$.

(2) Consider , what is W ?

$\Sigma = \Sigma_2$

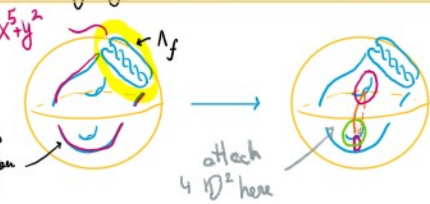


Thm : Let $\Lambda_f \subseteq (S^3, \xi_{st})$ be a max-tb Legendrian link which is smoothly the link of an isolated singularity $f \in \mathbb{C}\{x,y\}$. Then $\exists$ an arboreal Lagrangian skeleton $\mathbb{L}_{\tilde{f}}$ for the Weinstein 4-manifold $W(\Lambda_f)$ given by

$$\rightsquigarrow \quad \boxed{\mathbb{L}_{\tilde{f}} = M_{\tilde{f}} \cup_{\upsilon} \left(\bigcup_{i=1}^{\mu} \mathbb{D}^2\right)},$$

with υ the vanishing cycles of a real Morsification $\tilde{f}$. $\square$

Pictorially : $x^5 + y^2$



⊛

1. Examples
2. Discussion on Λ_f
3. Proof of thm.

Explicit examples of $W(\Lambda_f)$:

(1) For $f(x,y) = x + y^2$, smooth case,
Λ_f = unknot, $M_f = \mathbb{D}^2$, $\upsilon = \phi \Rightarrow \mathbb{L} = S^2$.
This is a skeleton for $W(\Lambda_0) \cong T^*S^2$.

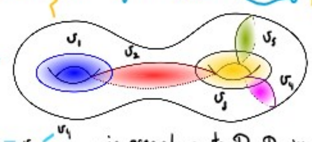
(2) For $f(x,y) = x^2 + y^2$, Λ_f = Hopf link, M_f = cylinder
and υ = {core cylinder} $\Rightarrow \mathbb{L} =$  $= S^2 \cup \mathbb{D}^2$.
This is skeleton for $W(\text{Hopf})$ = 2-plumbing T^*S^2.

(3) For $f(x,y) = x^3 + y^2$, Λ_f = trefoil, $M_f = \Sigma_{1,1}$ =
and $|\upsilon| = 2 \Rightarrow \mathbb{L} =$

(4) For $f(x,y) = x^{2n+1} + y^2$: $\Lambda_f = (2, 2n+1)$-torus knot
and $|\upsilon| = 2n \Rightarrow \mathbb{L} =$

(5) Let $f(x,y) = x^4 + xy^2 = x(x^3 + y^2)$
Λ_f = trefoil $\cup$ unknot, $M_f = \Sigma_{2,2}$ =

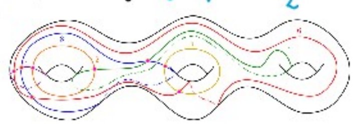
D_5 $\quad \mathbb{L} =$



Note $\upsilon_1 - \upsilon_2 - \upsilon_3 <^{\upsilon_4}_{\upsilon_5}$, in accordance to D_5 Dynkin.

(6) For $f(x,y) = x^3 + y^4$, $g(M_f) = \frac{(3-1)(4-1)}{2} = 3$

t $\quad \mathbb{L} =$



Note again, intersection according to mutation class of E_6 Dynkin quiver.

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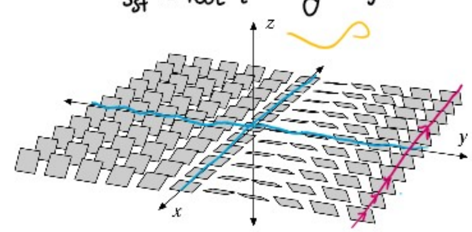
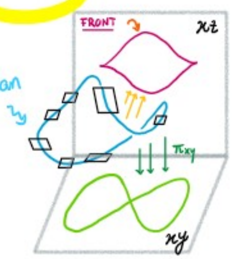
§2. LEGENDRIAN LINKS AND ISOLATED SINGULARITIES : LEGENDRIAN KNOTS in $\mathbb{R}^3$

Def^n: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field

$$\xi_{st} := \ker\{dz - y\,dx\}. \text{ It compactifies to } (\mathbb{S}^3, \xi_{st}).$$



Legendrian knot $\rightsquigarrow$



Def^n: A knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{LEGENDRIAN}}$ if $TK \subseteq \xi_{st}$.
A knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{TRANSVERSE}}$ if $TK \not\subseteq \xi_{st}$.

Examples of transverse and Legendrian links:

(i) Let $(S^3, \xi_{st}) \subseteq \mathbb{C}^2$, $\xi_{st} = TS^3 \cap i\,TS^3$, and $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ (a germ of) an isolated singularity at the origin.

Then $\underbrace{\{f=0\} \cap S^3_\varepsilon}_{\text{link of singularity}} \subseteq (S^3_\varepsilon, \xi_{st})$ is a transverse link.

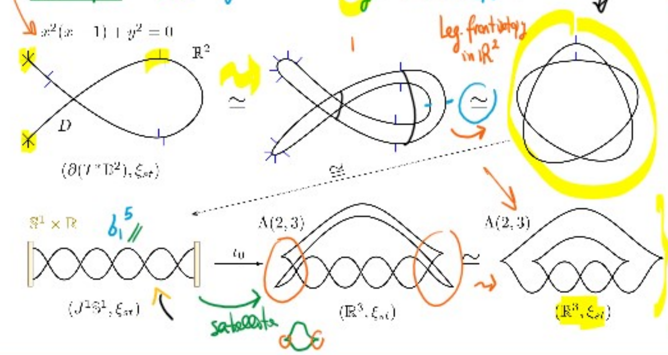
(ii) Any cooriented immersed plane curve $\gamma \subseteq \mathbb{R}^2$, possibly with cusps $(3/2)$ defines a Legendrian link $\Lambda_\gamma \subseteq (T^\infty \mathbb{R}^2, \xi_{st}) \cong (J^1S^1, \xi_{st})$.

~~satellite~~ $S^1 \subseteq J^1S^1$ to unknot in S^3 $\longrightarrow$ $\boxed{\Lambda_\gamma \subseteq (S^3, \xi_{st}) \text{ Legendrian}}$

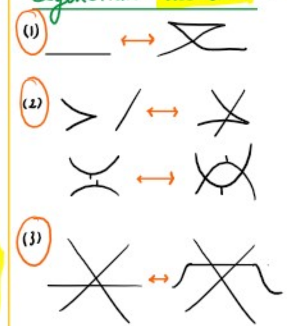
Front projections: since $TK \subseteq \ker\{dz - y\,dx\}$, we have $\boxed{y = \partial_x z(x)}$ on K.

$\Rightarrow$ the projection $\pi_{xz}(K)$ recovers K up to Legendrian isotopy. ← in $(\mathbb{R}^3, \xi_{st})$

Example: the trefoil knot $\Lambda_{x^3+y^2} \subseteq (S^3, \xi_{st})$:



Legendrian Reidemeisters:



Thm: Let $K \subseteq S^3$ be the link of an isolated singularity. ← positive smooth iterated cable

Etnyre / Tosun / LaFountain

(1) There exists a unique* transverse representative $T_K \subseteq (S^3, \xi_{st})$, with maximal self-linking number.

not true for other smooth K! $m(5_2)$ fails

(2) There exists a unique* Legendrian representative $\Lambda_K \subseteq (S^3, \xi_{st})$, with maximal Thurston-Bennequin number.

* up to trans. & Leg. isotopy!

An interesting construction: the "transverse push-off" $\tau(\Lambda)$ of a Legendrian link Λ is the transverse link $\tau(\Lambda)$ constructed as

(1) Same smooth type.

(2) Any two Λ_1, Λ_2 with $\tau(\Lambda_1) = \tau(\Lambda_2)$ differ by stab.

Def^n: Let $f \in \mathbb{C}\{x,y\}$ have an isolated singularity at the origin. The Legendrian link Λ_f of f is any max-tb Legendrian $\Lambda_f \subseteq (S^3, \xi_{st})$ such that $\tau(\Lambda_f)$ is transversely isotopic to the link of the singularity. ∎

By the previous theorem, Λ_f defines a unique Legendrian isotopy class.

Now, given f, how do we find Λ_f and a "good" front for it?

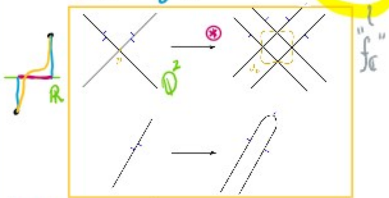
→ use the theory of real Morsifications of S. Gusein-Zade & N. A'Campo. In particular, the divides of an $\mathbb{R}$-Morsification!

→ the same thy actually gives a "Lagrangian skeleton" for the relative pair $(\mathbb{C}^2, \Lambda_f)$.

Sketch of Proof of the Lagrangian Skel. Thm.:

$\boxed{1^{st}}$ Let $\check{f} \in \mathbb{R}[x,y]$ be a $\mathbb{R}$-Morsification for $f \in \mathbb{C}[x,y]$, and D its divide. (first choose a real form)

Then Λ_D is such that $\tau(\Lambda_D) \cong \tau(\Lambda_f)$, so that D is a front for Λ_f.

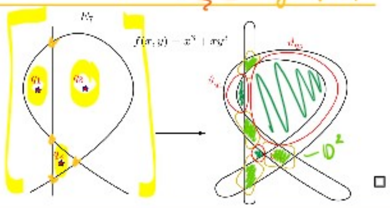


$\boxed{2^{nd}}$ Smooth the conormal lift of D inside $T^*\mathbb{R}^2$ to produce a Lagrangian filling L_D for $\Lambda_f \subseteq (\partial T^*\mathbb{R}^2)$. $\rightsquigarrow$ (see Goncharov's conj. surface)

$\boxed{3^{rd}}$ Read a basis of vanishing cycles from the divide D: contributions from

- Bounded regions • Crossings ⊛

Lagrangian 2-disks ending on L_D



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§ 3. Invariants associated to Λ_f (and thus to the singularity f)

Relative Invariants of $(\mathbb{C}^2, \Lambda_f)$

- geometrically: the space of embedded exact Lagrangian fillings.

- Algebraically: (1) Chekanov's DGA $A(\Lambda_f)$ and its augmentations $A(\Lambda_f) \longrightarrow \mathbb{R}$

(2) Categories of sheaves $Sh_\Lambda(\mathbb{R}^2)$.

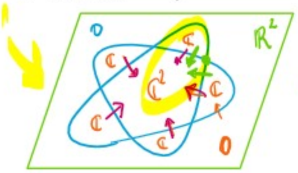
Absolute Invariants of $W(\Lambda_f)$

- geometrically: the space of embedded exact Lagrangians. (e.g. closed ones)

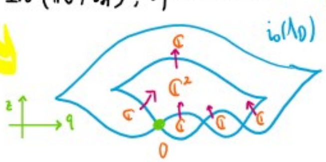
- Algebraically: "wrapped Fukaya categories" e.g. (dg-mod)-$A(\Lambda_f)$.

Down to Earth Computations: using FRONTS and SKELETA we can compute!

Consider $f(x,y) = x^3 + y^2$ then a divide is isotopic to:



In $(\mathbb{R}^3, \xi_{st})$, after satellite



Floer thy. $\longrightarrow$

$\{xyz+x+z=1\}$

Chekanov's DGA: $\mathbb{C}\langle a_1, a_2, x, y, z\rangle$
$\lvert a_1\rvert = \lvert a_2\rvert = 1, \quad \lvert x\rvert = \lvert y\rvert = \lvert z\rvert = 0$
$\partial a_1 = \partial a_2 = xyz + x + z + 1$
$\partial x = \partial y = \partial z = 0$

the "moduli space of Lagrangian fillings" is given by 5-tuple $(l_1, l_2, l_3, l_4, l_5)$ of lines in $\mathbb{C}^2$, cyclically ordered, with $l_i \neq l_{i+1}$ (mod 5). N.subsp. (this is mod $GL_2(\mathbb{C})$-action)

$\hookrightarrow$ this is an algebraic variety $X \rightsquigarrow$ RING OF FUNCTIONS is INTERESTING!

Connection to Fomin - Pylyavskyy - Shustin - Thurston:

$f(x,y)$ isolated singularity $\longrightarrow$ Legendrian link Λ_f $\longrightarrow$ moduli stack of Lagr. fillings $Aug(\Lambda_f)$ $\longrightarrow$ algebra of fct $\mathbb{C}[Aug(\Lambda_f)]$

choose $\downarrow$ $\mathbb{R}$-Morsif. $\check{f}$

$A\Gamma$-diagrams of Gusein-Zade and A'Campo divides

indep. of Morsif.

contact geom. ⇣ cluster alg.

might depend on Morsif. $\rightarrow$ L Lagrangian filling "$\in$"

one cluster in

thm (STZW-GSW) This is a cluster algebra.

Quiver $Q_{\check{f}}$

FPST: in some cases $\check{f}_1, \check{f}_2$ $\mathbb{R}$-Mors. of some f $\Updownarrow$ $Q_{\check{f}_1} \overset{⊛}{\sim} Q_{\check{f}_2}$

$\longrightarrow$ cluster algebra associated to (any!) $Q_{\check{f}}$

⊛ quiver mutation equivalence

APPLICATIONS : "INFINITELY MANY LAGRANGIAN FILLINGS" ← Northwestern talk G&P Seminar if interested

Let $f \in \mathbb{C}[x,y]$, $\Lambda_f \subseteq (S^3, \xi_{st})$ and $\mathbb{L}_{\tilde{f}}$ a Lagrangian Skeleton, as above.

idea : x^3+y^6 ← simple example with cluster algebra of infinite type

FIRST EXAMPLES KNOWN IN CONTACT GEOM.

⇒ intersection quiver of VC mutates

⇓

different clusters

⇓

different Lagrangian fillings.

Thm (2020) For $f(x,y) = x^a + y^b$, $a \geq 3$, $b \geq 6$ or $(4,4), (4,5)$, the Legendrian Λ_f has ∞'ly many Lagr. fillings. ∎

+ now we know "most" Leg. links are like that!

§4. ADE Conjecture (see also BCFG cases!) ← arXiv 2009.06737

Conj. (2020) Let $\Lambda \subseteq (S^3, \xi_{st})$ be a closure of a positive braid and Q_Λ connected*.

Then

(i) $\Lambda \cong \Lambda(A_n)$ and Λ has precisely $\frac{1}{n+2}\binom{2n+2}{n+1}$ Lagr. fill. //

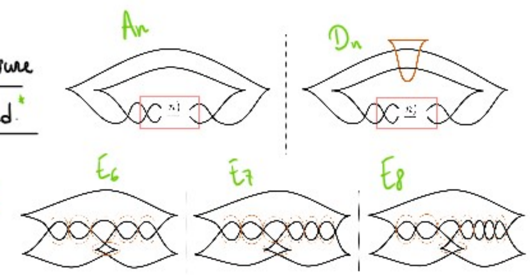
(ii) $\Lambda \cong \Lambda(D_n)$ and Λ has precisely $\frac{3n-2}{n}\binom{2n-2}{n-1}$ Lagr. fill. //

(iii) $\Lambda \cong \Lambda(E_6), \Lambda(E_7)$ or $\Lambda(E_8)$ with 833, 4160 and 25080 Lagr. fillings. //

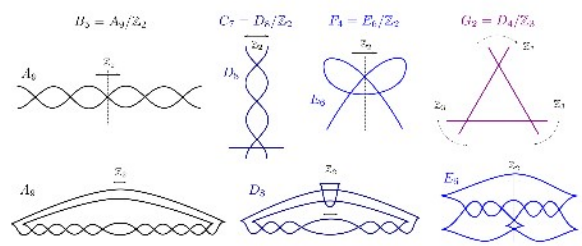
OR

{(iv) Λ has ∞'ly many Lagr. fillings.}

proven for positive braids (not ADE)



THE END



Thank you!