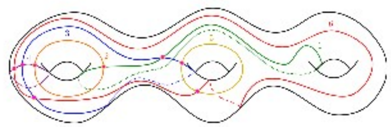
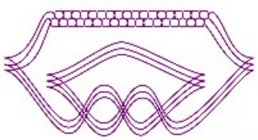
Legendrian Knots & Algebraic Structures



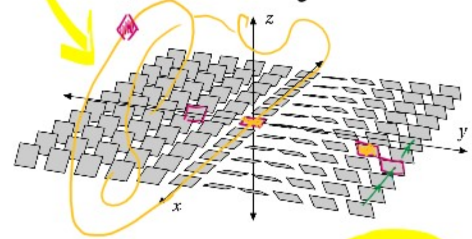


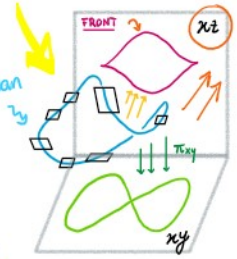
R. Casals (UC Davis) @ Dec 1

GRT at home Seminar

§1. INTRODUCTION : LEGENDRIAN KNOTS in $\mathbb{R}^3$

Def^n: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field $\xi_{st} := \mathrm{Ker}\{dz - y\,dx\}$. It compactifies to $(\mathbb{S}^3, \xi_{st})$. ∎





Def^n: A knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{LEGENDRIAN}}$ if $TK \subseteq \xi_{st}$. ∎

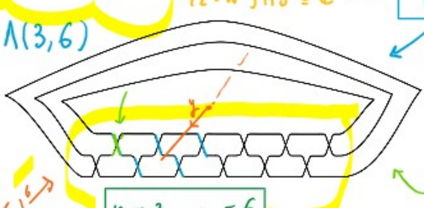
Front projections: since $TK \subseteq \mathrm{Ker}\{dz - y\,dx\}$, we have $\boxed{y = \partial_x z(x)}$ on K.

⇒ the projection $\pi_{xz}(K)$ recovers K up to Legendrian isotopy.

$\overset{!!}{\text{the front of } K}$

Two Examples: smoothly $\{z^n = w^m\} \cap S^3 \subseteq \mathbb{C}^2$ ⟵ $\boxed{\text{LEGENDRIAN TORUS LINKS}}$: $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$

$\Lambda(3,6)$

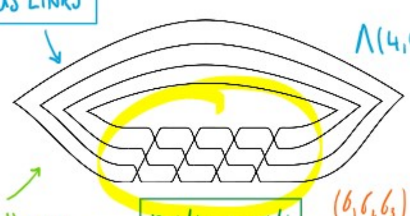


$(\sigma_1\sigma_2)^6$

$\boxed{n = 3,\ m = 6}$

max-tb rep.

$\Lambda(4,4)$

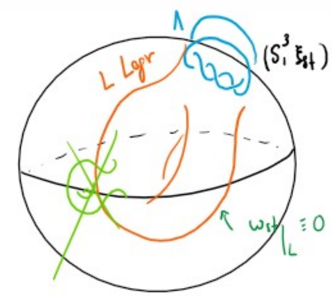


$(\sigma_1\sigma_2\sigma_3)^4$

$\boxed{n = 4,\ m = 4}$

LAGRANGIAN FILLINGS: let $\Lambda \subseteq (S^3, \xi_{st})$ be a Leg. link.

Def^n: A Lagrangian filling $L \subseteq (\mathbb{D}^4, \omega_{st})$ is an embedded exact Lagrangian surface in $\mathbb{D}^4$ with boundary $\partial L = \Lambda$ in $\partial\mathbb{D}^4 = S^3$. ∎



Salient Facts:

(1) A Λ might or might not have a Lagr. filling.

(2) If $\exists L$ filling Λ then $\boxed{g(L) = g_s(\Lambda)}$. different than smooth top.!

(3) (Eliashberg - Polterovich 1996) let $\Lambda = \Lambda_0$ be the max-tb standard unknot. Then $\exists!\ L$ filling (the flat disk) up to Hamiltonian isotopy.

(4) Lagr. fillings are the objects of $\mathcal{W}(\mathbb{C}^2, \Lambda)$, the wrapped Fukaya category stopped at Λ. (See also Sh_Λ.)

The augmentation variety $Aug(\Lambda(\beta))$

: For general $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ we need Floer thy.

· essentially nothing known about Aug

· moduli defined as maps

$\varepsilon : DGA(\Lambda) \longrightarrow R$ ring.

Let $\beta \in Br_n^+$ be a positive braid (word), and consider

$i \in [1, n-1]$, $\sigma_i \rightsquigarrow B_i(z) = \begin{pmatrix} 1 & & \cdots & & 0 \\ & \ddots & & & \vdots \\ \vdots & & \boxed{\begin{smallmatrix} 0 & 1 \\ 1 & z \end{smallmatrix}} & & \\ & & & \ddots & \\ 0 & & \cdots & & 1 \end{pmatrix} \in GL(n, \mathbb{C}[z])$.

elt. of $GL(n, \mathbb{C}[z_1, \dots, z_\ell])$

Def^n: For $\gamma = \sigma_{i_1} \cdot \ldots \cdot \sigma_{i_\ell}$ define $X_0(\gamma) := \{ (z_1, \dots, z_\ell) \in \mathbb{C}^\ell : B_{i_1}(z_1) \cdot \ldots \cdot B_{i_\ell}(z_\ell) \cdot w_0 \text{ is upper-triang.} \} \subseteq \mathbb{C}^\ell$.

Thm. (C.-Gorsky-Gorsky-Simental) The variety $X_0(\beta \cdot \Delta)$ is a smooth irreducible complete intersection of dim $\ell(\beta)$, and admits a toric T-action$^{(*)}$ such that:

$$\boxed{X_0(\beta\cdot\Delta)/_T \underset{\text{alg. var.}}{\cong} Aug(\Lambda(\beta))}.$$

$(*)$ $z_j \longrightarrow t_{w_j(i_j+1)} \cdot t^{-1}_{w_j(i_j)} z_j$

$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & {t_2}/{t_1} z \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$

cluster structures, holom. symp. structures, stratifications & cohom. computations $\longleftrightarrow$ geometrically build Lagr. fillings, Floer thy. functoriality: $\Lambda_1 < \Lambda_2$ then $Aug(\Lambda_1) \xrightarrow{\exists} Aug(\Lambda_2)$

An instance of geometry $\leftrightarrows$ algebra

: a Lagrangian filling $L_\tau \subseteq (\mathbb{R}^4, \omega_{st})$ can be constructed by choosing an ordering $\tau \in S_{\ell(\beta)}$ for the crossings of β:

Fact: For each such τ, $\exists\, T_\tau \in X_0(\beta\Delta)$ toric chart s.t. $T_\tau \cong (\mathbb{C}^\times)^{\ell(\beta)}$ and $(\mathbb{C}^\times)^{n-1}$-stable.

geometric question: How many Lag. fillings of Λ? $\longleftrightarrow$ algebraic question: How many toric charts?

Lag. loops for Λ $\longleftrightarrow$ automorphism (possibly permuting toric charts)

Example: Let us consider $(2,n)$-torus link, $\Lambda = \Lambda(\beta)$ with $\beta = \sigma_1^n \in Br_2^+$.

There are $n!$ ways to open crossings: Lagr. fillings in bijection with (321)-avoiding permutations.

By direct count $X_0(\beta\Delta)/_T$ is an A_{n-1}-cluster variety with $C_n := \frac{1}{n+1}\binom{2n}{n}$ toric charts.

in this case a bijection; not in general (not even $\beta = \sigma_1^{n-2}\sigma_2\sigma_1^2\sigma_2 \sim D_n$)

§ 2. 2020 Developments : the discovery of INFINITELY MANY LAGR. FILLINGS

Thm. (C.-Gao 20) The Legendrian torus links $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$ have infinitely many distinct Lagrangian fillings. [$\Lambda(3,6)$ has a $PSL_2(\mathbb{Z})$ worth and $\Lambda(4,4)$ a $M_{0,4}$]

if $n \geq 3$, $m \geq 6$ or $(4,4), (4,5)$

In fact, $\exists\, \infty$'ly many hyperbolic and satellites knots with this property!

↪ this is the first result that can access ∞'ly many toric charts & use them to distinguish L's!

(1) We show as well: if $\gamma < \beta$, i.e. γ is obtained from β by opening crossings then

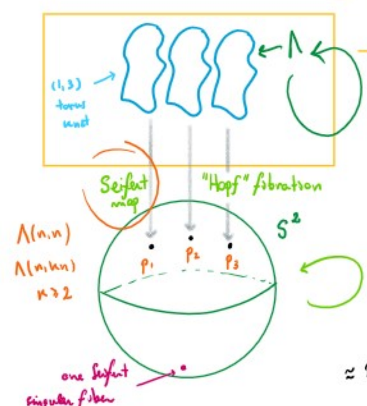
$\exists\, \infty$'ly fillings of γ (detected by $X_0(\gamma\Delta)/_T$) $\Rightarrow \exists\, \infty$'ly fillings of γ (in fact $[Aug(\gamma) \xhookrightarrow{\text{inj.}} Aug(\beta)]$)

(2) Currently $\beta = (\sigma_2\sigma_1\sigma_3\sigma_2)^4\sigma_3\sigma_1$ is the smallest positive braid with ∞'ly many fillings.

↪ aug. variety $X_0(\beta\Delta)/_T$ is of cluster $\tilde{A}_{1,1}$. $\tilde{A}_{1,1}$

Torus links $\Lambda(n,m)$ & LEGENDRIAN LOOPS

Consider the example of $\Lambda(3,6)$, as follows:



σ_1 acts as:

$\updownarrow$

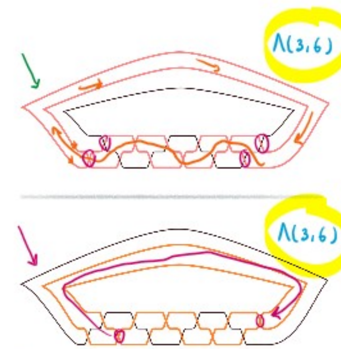
$PSL_2\mathbb{Z}$

$\updownarrow$

σ_2 acts as:

action of mapping class group $MCG(\mathbb{R}^2, p_1, p_2, p_3) \cong$

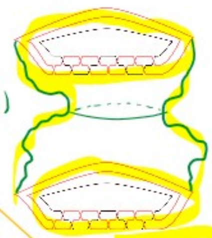
$\cong Br_3/_Z \cong PSL_2(\mathbb{Z}) = \langle \sigma_1, \sigma_2 \rangle$



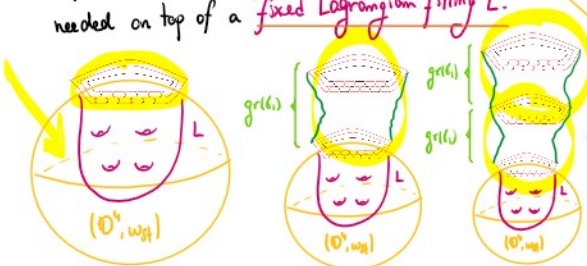
The candidate Lagrangian fillings :

* why am I doing $\Lambda(3,6)$ and not smaller ?

Step 1 : Consider the graph $gr(\sigma_1)$ of the Legendrian loop associated to σ_1.



Step 2 : Concatenate $gr(\sigma_1)$ as many times as needed on top of a fixed Lagrangian filling L.



$\Longrightarrow$

(1) L is smoothly isotopic to $L \cup gr(\sigma_1)$

(2) Are the Lagrangian fillings $L\cup gr(\sigma_1)^{\ell} \neq L\cup gr(\sigma_1)^{k}$? (if $\ell\neq k$)

THE SCHEME OF PROOF OF THE THEOREM : I will sketch it for $\Lambda(3,6)$. $\beta=(\sigma_1\sigma_2)^6$

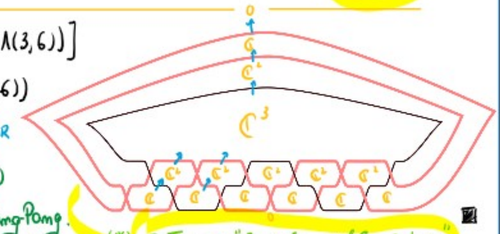
$\boxed{\text{STEP I}}$ Since σ_1 acts as an isotopy $\varphi^t_{\sigma_1}$, it acts on any invariant associated to $\Lambda(3,6)$. In particular, it gives an automorphism of $X_0(\beta\Delta)/T$.

$\boxed{\text{STEP II}}$ The augmentation moduli $Aug^{dec}(\Lambda(3,6))\ (\cong Sh^{fr}_{\Lambda}(\mathbb{R}^2))$ embeds as the open positroid cell in $Gr(3,9)$. Consider action on $\mathbb{C}[Gr(3,9)\setminus \text{anticanonical}]$.

$\boxed{\text{STEP III}}$ Show that $\exists$ Plücker fct $\in \mathbb{C}[Aug(\Lambda(3,6))]$ such that the group $\langle\sigma_1,\sigma_2\rangle \in \pi_1(Lag(3,6))$ acts faithfully in the orbit. THIS REQUIRES A FAIR (*) AMOUNT OF UNDERSTANDING OF σ_1,σ_2 & $Gr(3,9)$ $\longrightarrow$ for $Gr(3,9)$

we use G. Kuperberg's SL_3-web basis & Ping-Pong. (algebra helped geometry !)

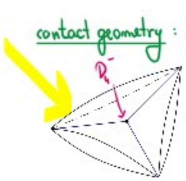
(*) C. Fraser's "Br. gp. Symm. of Gr. clust. alg."



§3. Weave Calculus

with E. Zaslow (arXiv 2007: 04943 ~ 114pp), also w/ Gorsky + Simental (soon)

contact geometry : Slicing of the D_4^- singularity yields an exact Lagr. cob. $\sigma_i\sigma_i \longrightarrow \sigma_i$.

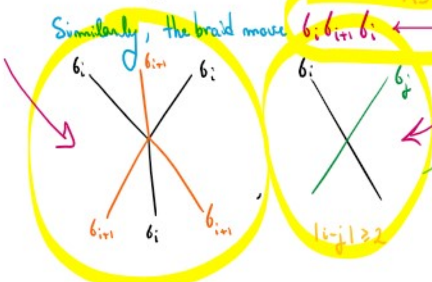


we represent this diagramatically with a trivalent vertex $\rightsquigarrow$

the exact Logr. cob. is the 2-fold branched cover : vertex = branch

Similarly, the braid move $\sigma_i\sigma_{i+1}\sigma_i \longleftrightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}$ is allowed (as is $\sigma_i\sigma_j \leftrightarrow \sigma_j\sigma_i$, $|i-j|\geq 2$)

R3

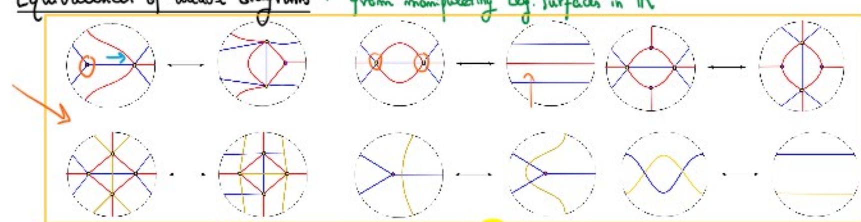


diagrammatic rep.^{n} for Lagrangian cobordisms } ALL DICTATED BY FLOER THY & LEG. LINKS !

equivalences ?

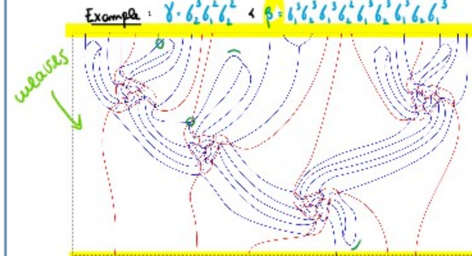
yield correspondences

Equivalences of weave diagrams : from manipulating Leg. surfaces in $\mathbb{R}^5$



Example : $\gamma = \sigma_2^3\sigma_1^2\sigma_2^2 < \beta = \sigma_1^3\sigma_2^3\sigma_1^3\sigma_2^2\sigma_1^3\sigma_2^3\sigma_1^3\sigma_2\sigma_1^3$

weaves



(1) we can add cups $\overset{\sigma_i\ \ \sigma_i}{\cup}$ and caps $\underset{\sigma_i\ \ \sigma_i}{\cap}$, then fillings become immersed.

(2) these 2 Lagrangian cobordisms are DISTINCT :

(Here you may see C_n appearing for $\beta=\sigma_1^n$ as before !)

Flag moduli & Correspondences:

Def: the flag moduli of a weave is the moduli of flags which are transverse wrt the weave.

Thm ('20): (i) A weave $\Sigma : \beta_1 \longrightarrow \beta_2$ yields a correspondence of alg. var.

$$\begin{tikzcd} & \mathcal{Fl}(\Sigma) \arrow[dl] \arrow[dr] & \\ X_0(\beta_1) & & X_0(\beta_2) \end{tikzcd}$$

(ii) $\mathcal{Fl}(\Sigma)$ yields an INJECTIVE map

$$\mathbb{C}^a \times (\mathbb{C}^*)^b \times X_0(\beta_1) \longrightarrow X_0(\beta_2)$$

if Σ has a cups, b trivalent vertices and No caps.

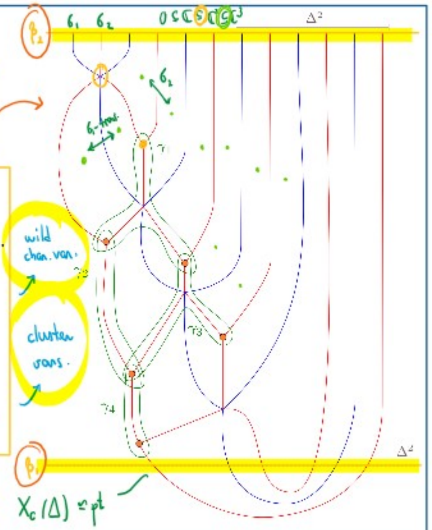
stratifications

holom. Darboux chart

study Lagr. fillings diagram.

wild char. var.

cluster vars.



Thm: (C.-Zaslow Spring '20) There exist Legendrian links $\Lambda \subseteq (\mathbb{S}^3, \xi_{st})$ for which the surgery strategy can be implemented. In particular, with a quiver Q_Λ of infinite mutation type ($\Longrightarrow$ ∞'ly many Lagr. fillings)

↳ PROOF USES NEW DIAGRAMMATIC CALCULUS, this is an example: "Leg. weaves"

§4. ADE Conjecture

(see also BCFG cases!) ⟵ arXiv 2009.06737

Conj. (2020) Let $\Lambda \subseteq (\mathbb{S}^3, \xi_{st})$ be a closure of a positive braid and Q_Λ connected.*

Then

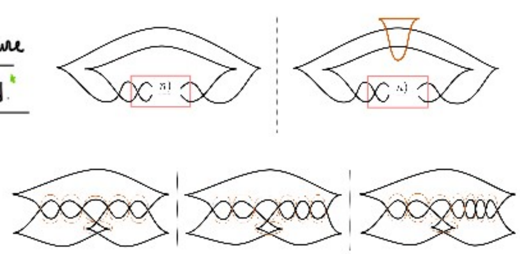
A (i) $\Lambda \cong \Lambda(A_n)$ and Λ has precisely $\frac{1}{n+2}\binom{2n+2}{n+1}$ Lagr. fill.

D (ii) $\Lambda \cong \Lambda(D_n)$ and Λ has precisely $\frac{3n-2}{n}\binom{2n-2}{n-1}$ Lagr. fill.

E (iii) $\Lambda \cong \Lambda(E_6), \Lambda(E_7)$ or $\Lambda(E_8)$ with 833, 4160 and 25080 Lagr. fillings.

OR

(iv) Λ has ∞'ly many Lagr. fillings.



SUMMARY OF TECHNIQUES:

they all prove ∞'ly Lagr. fill.

Direct methods: (C.-Gao '20)

- Yield the strongest results (e.g. $PSL_2\mathbb{Z}$)
- they are hard to prove ⇝ categorical faithfulness? (see C. Fraser & B. Keller)

Lagr. skel & weaves: (C.-Zaslow Spring '20)

- only using the mutation class of Q_Λ
- the diagrams can get seriously complicated

rel^n to Soergel calc. & Charact. varieties

work-in-progress by J. Hughes for weaves in the D_n-case

C.-Gorsky-Gorsky-Simental

Floer theory: (C.-Ng Fall '20)

- the only method that can tackle links which are not positive.
- required developing the theory / $\mathbb{Z}$ → work-in-progress by S. Rubin for a char 2 argument

Cluster algebras: (Gao-Shen-Weng Fall '20)

- immediate if the DT transformation has ∞ order
- the case of non-positive links is a mystery. (currently only $\mathbb{Z}_2$) → Augmentation stacks are cluster?

THE END

Thank you!