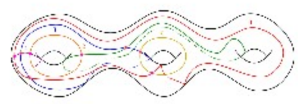
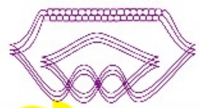
Legendrian Knots & Lagrangian fillings



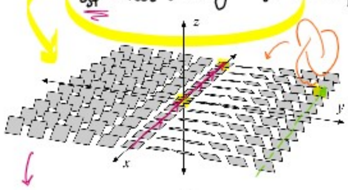


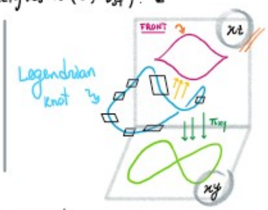
R. Casals (UC Davis) @
IBS-CGP "Legendrians, cluster algebras & MS"

(1) 2001.01334 w/ H. Gao
(2) 2007.04943 w/ E. Zaslow
2009.06737 (3)
2101.02318 (4) w/ L. Ng
(5) 2012.06931 w/ E. Gorsky, M. Gorsky, J. Simental

§1. INTRODUCTION : LEGENDRIAN KNOTS in $\mathbb{R}^3$

Def^n: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field $\xi_{st} := \ker\{dz - y\,dx\}$. It compactifies to $(\mathbb{S}^3, \xi_{st})$. ∎





Def^n: A knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{LEGENDRIAN}}$ if $TK \subseteq \xi_{st}$. ∎

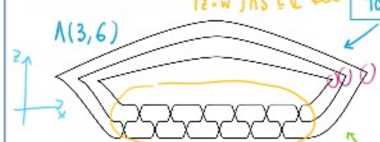
Front projections: since $TK \subseteq \ker\{dz - y\,dx\}$, we have $\boxed{y = \partial_x z(x)}$ on K.

⇒ the projection $\pi_{xz}(K)$ recovers K up to Legendrian isotopy.
‖
the front of K

Two Examples: $\boxed{\text{LEGENDRIAN TORUS LINKS}}$: $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$

smoothly $\{z^n = w^m\} \cap S^3 \subseteq \mathbb{C}^2$

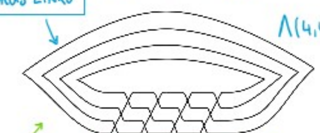
$\Lambda(3,6)$



$\boxed{n = 3,\ m = 6}$

max-tb rep.

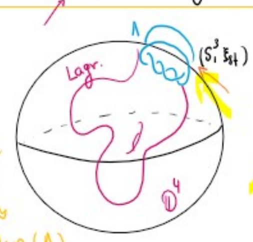
$\Lambda(4,4)$



$\boxed{n = 4,\ m = 4}$

LAGRANGIAN FILLINGS : let $\Lambda \subseteq (S^3, \xi_{st})$ be a Leg. link.

Def^n: A Lagrangian filling $L \subseteq (\mathbb{D}^4, \omega_{st})$ is an embedded exact Lagrangian surface in $\mathbb{D}^4$ with boundary $\partial L = \Lambda$ in $\partial\mathbb{D}^4 = S^3$. ∎



$\mathrm{Aug}(\Lambda)$

Salient Facts:

(1) A Λ might or might not have a Lagr. filling.

(2) If $\exists L$ filling Λ then $\boxed{g(L) = g_s(\Lambda)}$.
different than smooth top.!

(3) (Eliashberg-Polterovich 1996) let $\Lambda = \Lambda_0$ be the max-tb standard unknot. Then $\exists!\, L$ filling (the flat disk) up to Hamiltonian isotopy.

(4) Lagr. fillings are the objects of $\mathcal{W}(\mathbb{C}^2, \Lambda)$, the wrapped Fukaya category stopped at Λ. (See also Sh_Λ.)

THE MODULI OF LAGRANGIAN FILLINGS — the mirror of the "LG Model" $(\mathbb{C}^2, \Lambda)$

bdy condition at ∞

Let $\Lambda \subseteq (S^3, \xi_{st})$ be Legendrian. Two $\boxed{\text{legendrian isotopy invariants}}$ (of some).

FLOER THY

Y. Chekanov '00, Ekholm-Etnyre-Sullivan, Ng, and more

$\boxed{\text{LEGENDRIAN CONTACT DGA } A_\Lambda}$

(can be enhanced to a category Aug_+)

Let $L \subseteq (\mathbb{D}^4, \lambda_{st})$ be a Lagrangian filling, induces $\varepsilon_L : A_\Lambda \longrightarrow \mathbb{Z}[H_1(L)]$ "augmentation".

$\boxed{\text{Augmentation variety } \mathrm{Aug}}$ → cluster variety (if $\Lambda = \Lambda(\beta)$)

affine variety

holomorphic symplectic [C.-Gorsky^2-Sim.]

SHEAF THY

$\boxed{\text{SHEAVES on } \mathbb{R}^2 \text{ MICROLOCALLY SUPPORTED at } \Lambda}$

Sato (Hörmander), Kashiwara-Schapira, Guillermou, Shende-Treumann-Zaslow

Let $L \subseteq (\mathbb{D}^4, \lambda_{st})$ be a Lagrangian filling, induces an object in $Sh_\Lambda(\mathbb{R}^2)$.

$\boxed{\text{"moduli of objects" in } Sh_\Lambda(\mathbb{R}^2)}$ → cluster structures [STZW] (∀ Λ(β)) → holom. symp. [Bo, ST]

⟶ TWO INCARNATIONS of the moduli of fillings

§2. 2020 Developments : the discovery of INFINITELY MANY LAGR. FILLINGS

See H. Gao's talk!

Thm. (C.-Gao 20) The Legendrian torus links $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$ have infinitely many distinct Lagrangian fillings. [$\Lambda(3,6)$ has a $PSL_2(\mathbb{Z})$ worth and $\Lambda(4,4)$ a $M_{0,4}$]
In fact, $\exists$ ∞'ly many hyperbolic and satellites knots with this property!

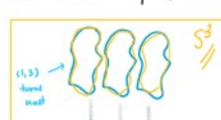
Cor: There exists an abundance of Stein 4-manifolds W^4 homotopic to the 2-sphere $\mathbb{S}^2$ with ∞'ly many Lagrangian surfaces of genus g (and no Lagr. surface of genus $< g$). ∎

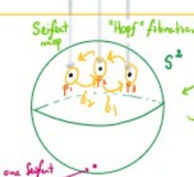
Rmk: the Thm. is stronger than this, constructing a $PSL_2(\mathbb{Z})$ subgroup of the Lagrangian concordance monoid. Also, all the filling will be explicit!

acting faithfully on Lagr. fillings

Torus links $\Lambda(n,m)$ & LEGENDRIAN LOOPS

Consider the example of $\Lambda(3,6)$, as follows:

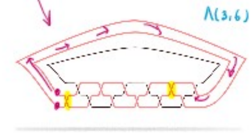




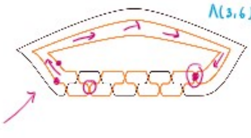
action of mapping class group $MCG(\mathbb{R}^2, p_1, p_2, p_3) \cong$
$\cong Br_3/\mathbb{Z} \cong PSL_2(\mathbb{Z}) = \langle \sigma_1, \sigma_2 \rangle$

$\boxed{\sigma_1 \text{ acts as :}}$ → $PSL_2\mathbb{Z}$ ← $\boxed{\sigma_2 \text{ acts as :}}$

$\Lambda(3,6)$



$\Lambda(3,6)$

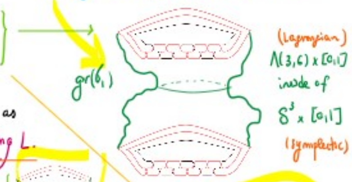


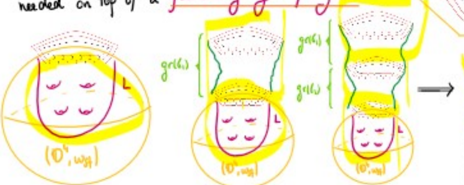
The candidate Lagrangian fillings :

* why am I doing $\Lambda(3,6)$ and not smaller ?

Step 1 : Consider the graph $gr(\sigma_1)$ of the Legendrian loop associated to σ_1.

Step 2 : Concatenate $gr(\sigma_1)$ as many times as needed on top of a fixed Lagrangian filling L.





(1) L is smoothly isotopic to $L\cup gr(\sigma_1)$

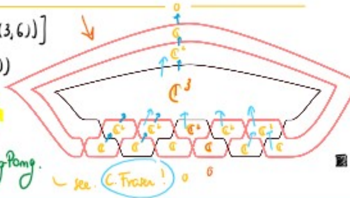
(2) Are the Lagrangian fillings $L\cup gr(\sigma_1)^\ell \neq L\cup gr(\sigma_1)^k$? (if $\ell\neq k$)

THE SCHEME OF PROOF OF THE THEOREM : I will sketch it for $\Lambda(3,6)$.

STEP I : Since σ_1 acts as an isotopy $\varphi^t_{\sigma_1}$, it acts on any invariant associated to $\Lambda(3,6)$. In particular, it gives a self-equiv. on Sh_Λ.

STEP II : The moduli $\mathcal{M}^{fr}(\Lambda(3,6))$ of framed sheaves embeds as the open positroid cell in $Gr(3,9)$. Consider action on $\mathbb{C}[Gr(3,9)\setminus \text{anticanonical}]$.

STEP III : Show that $\exists$ Plücker fct $\in \mathbb{C}[\mathcal{M}^{fr}(\Lambda(3,6))]$ such that the group $\langle\sigma_1,\sigma_2\rangle\in\pi_1(Lag(3,6))$ acts faithfully in the orbit. THIS REQUIRES A FAIR AMOUNT OF UNDERSTANDING OF σ_1,σ_2 & $Gr(3,9)$ $\longrightarrow$ for $Gr(3,9)$ we use G. Kuperberg's SL_3-web basis & Ping-Pong.



see C. Fraser !

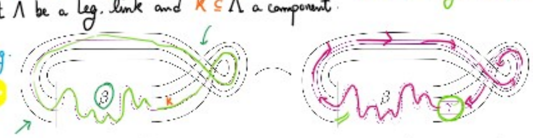
§ 3. Two more discoveries : ϑ-loops & Weave methods (+DT by H. Gao & collab.)

with L. Ng (Fall '20) — with E. Zaslow (Spring '20)

∞'ly many fillings I : ϑ-loops $\longleftarrow$ generalizes the loops constructed above to many more links

Let Λ be a Leg. link and $K\subseteq\Lambda$ a component.

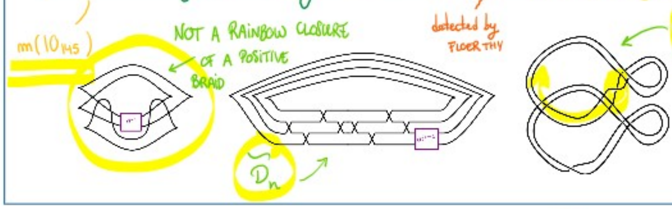
E.g.



Then satellite $\gamma=\sigma_1^n$ along K. This creates a Legendrian loop by moving the purple box with γ inside around. $\longrightarrow$ we call them ϑ-loops.

Thm : (C.-Ng : see next talk!) Let Λ be any of the Legendrians links below. (among many !) Then the ϑ-loops associated to the purple box monodromy have ∞ order when acting on the augmentation $\varepsilon_L:\mathcal{A}_\Lambda\longrightarrow\mathbb{Z}[H_1(L,\mathbb{Z})]$ associated to a certain Lagrangian filling of Λ. (In particular, ∞'ly many fillings !)

detected by FLOER THY



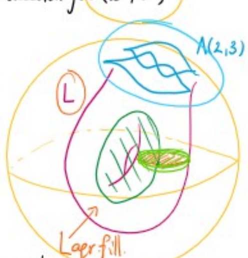
∞'ly many fillings II : Lagrangian skeleta & Weaves

A geometrization of cluster algebras : a Lagr. skeleton for $(\mathbb{D}^4,\Lambda)$ — Weinstein pair

STRATEGY : Suppose $L\subseteq(\mathbb{D}^4,\omega_{st})$ is a Lagrangian filling and $\exists$ Lagr. 2-disks $\mathbb{D}^2\subseteq\mathbb{D}^4\setminus L$ with boundary in L. Then we can SURGER THE $\mathbb{D}^2$'s.

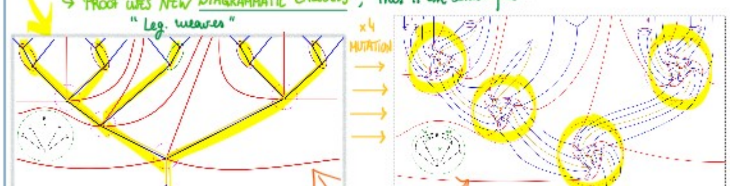
(a) **Thm.** (C'20) : This skeleton exists if Λ is a max-tb algebraic link.

(b) Still, iterating surgeries typically creates immersed curves $\longrightarrow$ hard to find $\mathbb{D}^2$'s to surger !



Thm : (C.-Zaslow Spring '20) There exist Legendrian links $\Lambda\subseteq(\mathbb{S}^3,\xi_{st})$ for which the surgery strategy can be implemented. In particular, with a quiver Q_Λ of infinite mutation type ($\Longrightarrow$ ∞'ly many Lagr. fillings)

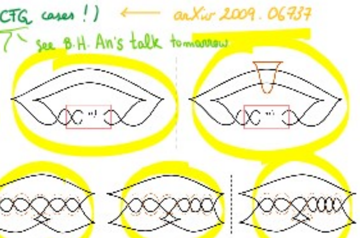
↳ PROOF USES NEW DIAGRAMMATIC CALCULUS, this is an example : "Leg. weaves"



§ 4. ADE Conjecture (see also BCTG cases !) $\longleftarrow$ arXiv 2009.06737

see B.H. An's talk tomorrow

Conj. (2020) Let $\Lambda\subseteq(\mathbb{S}^3,\xi_{st})$ be a closure of a positive braid and Q_Λ connected. Then



A (i) $\Lambda\cong\Lambda(A_n)$ and Λ has precisely $\longrightarrow \frac{1}{n+2}\cdot\binom{2n+2}{n+1}$ Lagr. fill.

D (ii) $\Lambda\cong\Lambda(D_n)$ and Λ has precisely * $\longrightarrow \frac{3n-2}{n}\cdot\binom{2n-2}{n-1}$ Lagr. fill.

E (iii) $\Lambda\cong\Lambda(E_6)$, $\Lambda(E_7)$ or $\Lambda(E_8)$ with 833, 4160 and 25080 Lagr. fillings.

OR

(iv) Λ has ∞'ly many Lagr. fillings.

(* Constructive bound at least 1 filling per cluster — J. Hughes (D type) — B.H. An - Y. Bae - E. Lee (ADE type))

SUMMARY OF TECHNIQUES : they all prove ∞'ly Lagr. fill.

Direct methods : (C.-Gao '20)
- Yield the strongest results (e.g. $PSL_2\mathbb{Z}$)
- they are hard to prove $\rightsquigarrow$ categorical faithfulness ? (see C. Fraser & B. Keller)

Lagr. skel & weaves : (C.-Zaslow Spring '20)
- only using the mutation class of Q_Λ , is it general ?
- the diagrams can get seriously complicated — rel^n to Soergel calc. & Charact. varieties — J. Hughes for weaves in the D_n-case — can always mutate !

Floer theory : (C.-Ng Fall '20)
- the only method that can tackle links which are not positive.
- required developing the theory / $\mathbb{Z}$ $\rightarrow$ work-in-progress by S. Rubin for a char 2 argument

Cluster algebras : (Gao-Shen-Weng Fall '20)
- immediate if the DT transformation has ∞ order
- the case of non-positive links is a mystery. $\rightarrow$ Augmentation stacks are cluster ? Hol. Sympl. ?

Future directions (of many!)

1. Let $\Lambda \subseteq (S^3, \xi_{st})$ be such that $Aug(\Lambda)$ cluster.

Consider $\mathcal{E} : \{\text{Lagr. fillings of } \Lambda\} \longrightarrow \{\text{clusters of } Aug\}$.

$L \longmapsto \mathcal{E}_L$

exc. surjective in finite type

Currently no examples where INJ. or SURJ. of $\mathcal{E}$ known.

2. Geography & Botany: study the map

$$\left\{\begin{matrix}\Lambda \text{ Legendrian link} \\ \text{in } (M,\xi)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{cluster} \\ \text{algebras}\end{matrix}\right\}$$

$$\Lambda \longmapsto \mathbb{C}[Aug(\Lambda)]$$

with $Aug(\Lambda)$ cluster

- Double Bruhat cells ✓
- Finite type ✓
- Affine type ?
- Quantum cluster alg. ?

3. Geometrization of algebraic features

(i) $\mathbb{C}[Aug(\Lambda)]$ cluster algebra, $f \in Mod(\mathbb{C}[Aug(\Lambda)])$ — cluster modular group

Is f induced by Legendrian isotopy ? — possibly more general ?

braid symmetry on $Gr(k,n)$ [CG]

DT^2 [GSW] (sometimes!)

Coxeter mutation [ABL]

BASIS !?

(ii) The q-parameter ?

(iii) Cluster A- & X-coord. in Aug & Sheaves ?

4. Develop & Standardize Connections to:

(i) Homological mirror symmetry — can we recover $\Lambda = \partial\mathbb{C}^2$ from $Aug(\Lambda)$?

(ii) Exact WKB analysis — see T. Kuwagaki's talk, also Iwaki + Nakanishi

WKB lines as Morse flow trees

(See also "spectral networks")

(iii) what is "holomorphic symplectic" topology ? — K_2-Lagr. in $Aug(\Lambda)$?

(iv) Singularities of plane curves — [FPST], R-Morf., natural cluster algebra

THE END

Thank you !