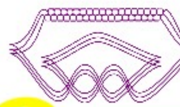
# Legendrian Knots & Lagrangian fillings



(1) 2001.01334 w/ H. Gao

(2) 2007.04943 w/ E. Zaslow


R. Casals (UC Davis) @

MSU "Geometry & Topology Seminar"

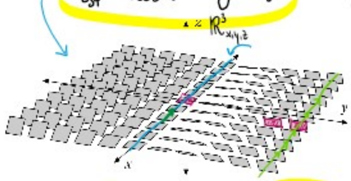

2009.06737 (3)

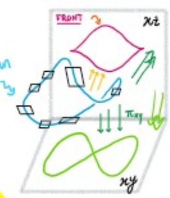
2101.02318 (4) w/ L. Ng

(5) 2012.06931 w/ E. Gorsky, M. Gorsky, J. Simental

## §1. INTRODUCTION : LEGENDRIAN KNOTS in $\mathbb{R}^3$

Def$^n$: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field $\xi_{st} := \ker\{dz - y\,dx\}$. It compactifies to $(\mathbb{S}^3, \xi_{st})$. ∎




Def$^n$: A Knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{LEGENDRIAN}}$ if $TK \subseteq \xi_{st}$. ∎

**Front projections**: since $TK \subseteq \ker\{dz - y\,dx\}$, we have $\boxed{y = \partial_x z(x)}$ on K.

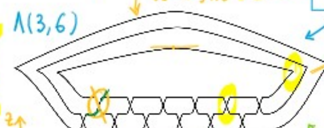
$\Rightarrow$ the projection $\pi_{xz}(K)$ recovers K up to Legendrian isotopy.

the front of K

Two Examples: $\boxed{\text{LEGENDRIAN TORUS LINKS}}$ : $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$

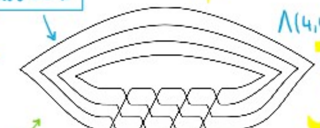
smoothly $\{z^n = w^m\} \cap S^3 \subseteq \mathbb{C}^2$

$\Lambda(3,6)$



$\boxed{n=3,\ m=6}$

$\Lambda(4,4)$

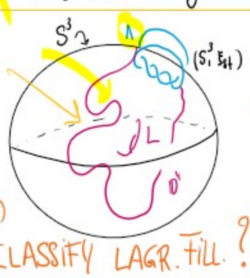


$\boxed{n=4,\ m=4}$

max-tb rep.

## LAGRANGIAN FILLINGS : let $\Lambda \subseteq (S^3, \xi_{st})$ be a Leg. link.

Def$^n$: A Lagrangian filling $L \subseteq (\mathbb{D}^4, \omega_{st})$ is an embedded exact Lagrangian surface in $\mathbb{D}^4$ with boundary $\partial L = \Lambda$ in $\partial\mathbb{D}^4 = S^3$. ∎




CLASSIFY LAGR. FILL. ?

Salient Facts:

(1) A $\Lambda$ might or might not have a Lagr. filling.

(2) If $\exists L$ filling $\Lambda$ then $\boxed{g(L) = g_s(\Lambda)}$ different than smooth top.!

(3) (Eliashberg-Polterovich 1996) let $\Lambda = \Lambda_0$ be the max-tb standard unknot. Then $\exists!$ L filling (the flat disk) up to Hamiltonian isotopy.

(4) Lagr. fillings are the objects of $\mathcal{W}(\mathbb{C}^2, \Lambda)$, the wrapped Fukaya category stopped at $\Lambda$. (See also $Sh_\Lambda$.)

## THE MODULI OF LAGRANGIAN FILLINGS — the mirror of the "LG Model" $(\mathbb{C}^2, \Lambda)$

bdy condition at ∞

Let $\Lambda \subseteq (S^3, \xi_{st})$ be Legendrian. A $\boxed{\text{Legendrian isotopy invariant}}$ (of some).

Y. Chekanov '00, Ekholm-Etnyre-Sullivan, Ng, and more

$\boxed{\text{LEGENDRIAN CONTACT DGA } \mathcal{A}_\Lambda}$

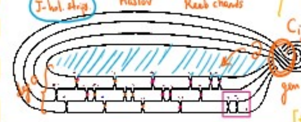
FLOER THY

(can be enhanced to a category $Aug_+$)

Two facts: $\partial C_{ij} = \boxed{\prod_{\text{6 cusps}} P_6 + \text{Id}} = 0$ if $\beta^+$ positive braid

(1) Differential Graded Algebra $\mathcal{A}_\Lambda$

J-hol. strips — Maslov — Reeb chords




[EHK, C.-Ng]

Let $L \subseteq (\mathbb{D}^4, \lambda_{st})$ be a Lagrangian filling, induces $\varepsilon_L : \mathcal{A}_\Lambda \longrightarrow \mathbb{Z}[H_1(L)]$ "augmentation".

(*) $\boxed{\text{Augmentation variety Aug}}$ — cluster variety (if $\Lambda = \Lambda(\beta)$)

affine variety

holomorphic symplectic [C.-Gorsky$^2$-Sim.]

(2) $\Lambda_1 \geq \Lambda_2$


induces explicit map $\mathcal{A}_{\Lambda_1} \longrightarrow \mathcal{A}_{\Lambda_2}$

## §2. 2020 Developments : the discovery of INFINITELY MANY LAGR. FILLINGS

Thm. (C.-Gao 20) The Legendrian torus links $\Lambda(n,m) \subseteq (\mathbb{S}^3, \xi_{st})$ have infinitely many distinct Lagrangian fillings. [$\Lambda(3,6)$ has a $PSL_2(\mathbb{Z})$ worth and $\Lambda(4,4)$ a $M_{0,4}$]

In fact, $\exists$ ∞'ly many hyperbolic and satellites knots with this property!

Cor: There exists an abundance of Stein 4-manifolds $W^4$ homotopic to the 2-sphere $\mathbb{S}^2$ with ∞'ly many Lagrangian surfaces of genus g (and no Lagr. surface of genus < g). ∎

Subsequently: Legendrian weaves (C.-Zaslow), cluster structures (Gao-Shen-Weng), Coxeter transformation & realizability (Hughes & An-Bae-Lee), Cat Sheldon (C.), Holomorphic symp. str. (C.-Gorsky$^2$-Simental) and more. All these use sheaves and/or cluster algebras and only apply to $\boxed{\beta_+}$.

## Today's Results : Floer thy in action

CG20: [if $\Lambda_1$ has ∞ fillings and $\Lambda_1 \geq \Lambda_2$, then $\Lambda_2$ has ∞ fill.] ⤳ Aug var

with L. Ng

Thm A: (Existence) There exists many Leg. links $\Lambda \subseteq (S^3, \xi_{st})$ which admit:

(i) A Legendrian loop $\vartheta_t \in \text{Cont}(S^3, \xi_{st})$, $\vartheta_0 = id$, $\vartheta_1(\Lambda) = \Lambda$,

(ii) A Lagrangian filling $L \subseteq (\mathbb{D}^4, \lambda_{st}) \rightsquigarrow \varepsilon_L$ aug.

such that $A(\vartheta) \in \text{Aut}(\mathcal{A}_\Lambda)$ satisfies

$$(A(\vartheta)^n)^*(\varepsilon_L) = (A(\vartheta)^m)^*(\varepsilon_L) \text{ iff } n = m.$$

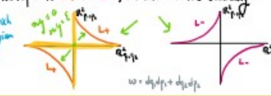
- geometrically, this implies ∞'ly Hamiltonian isot. classes of Lagrangian fillings.
- cool extra, they are the SMALLEST known to date (and not positive braids!)

FIGURE 1

Thm B: (Uniqueness) Let $\Lambda_{Hopf} \subseteq (S^3, \xi_{st})$ be the max-tb Legendrian Hopf link. Then any embedded Lagrangian filling $L \subseteq (\mathbb{D}^4, \lambda_{st})$ of $\Lambda_{Hopf}$ is Hamiltonian isotopic to $L_+$ or $L_-$, relative to the boundary.

in progr. C.-O.-R.

(*) $L_\pm$ is the $\pm$-Polterovich surgery of the Lagrangian coordinate planes $\langle q_1, q_2 \rangle$, $\langle p_1, p_2 \rangle$.




- complete classification for all unknots & Hopf links. (of Lagr. fillings up to Ham.)
- $Aug(\Lambda_{Hopf})$ is an $A_1$-cluster variety, this shows clusters ⟷ fillings.

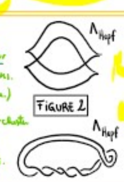


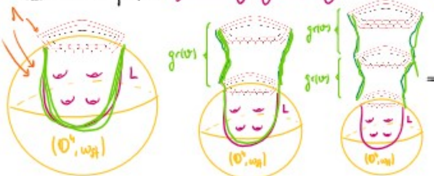

FIGURE 2

## The candidate Lagrangian fillings :

$\vartheta_0 = id$ , $\vartheta_1(\Lambda) = \Lambda$

**Step 1** : Consider the graph $gr(\vartheta)$ of the Legendrian loop associated to $\vartheta$.


**Step 2** : Concatenate $gr(\vartheta)$ as many times as needed on top of a fixed Lagrangian filling L.



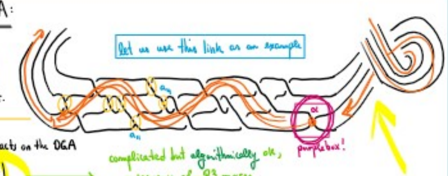

$\Longrightarrow$

(1) L is smoothly isotopic to $L \cup gr(\vartheta)$

(2) Are the Lagrangian fillings $L \cup gr(\sigma_1)^{\ell} \neq L \cup gr(\sigma_1)^{k}$ ? (if $\ell \neq k$)

## THE SCHEME OF PROOF OF THEOREM A:

STEP I: Build a Lagrangian filling L by EHK opening crossings until $\Delta^2$ is left.




STEP II: Study how the purple box monodromy acts on the DGA

e.g. $A(\vartheta)\cdot\begin{pmatrix} a_{11} \\ a_9 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_9 \end{pmatrix}$ ⟶ complicated but algorithmically ok, a sequence of R3 moves

STEP III: Compute the augmentation $\varepsilon_L : A_\Lambda \longrightarrow \mathbb{Z}[H_1(L,\mathbb{Z})]$ induced by L ← (technical, but doable)

and then show that the pull-backs of $\varepsilon_L$ by $A(\vartheta)^m$ are NOT $GL_{b_1(L)}(\mathbb{Z})$-equivalent. ← this is the hard part

E.g. $A(\vartheta)^k \varepsilon_L \begin{pmatrix} a_{11} \\ a_9 \end{pmatrix} = \begin{pmatrix} s_{11} & 1 \\ s_9 & 0 \end{pmatrix} \cdot M_1^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where

$M_1 = \begin{pmatrix} -s_9 s_{12}^2 s_{13}/s_{10}s_{11} - s_9 s_{16} & s_{12}s_{13}/s_{11} \\ s_9 s_{12}^2 s_{13}/s_{10} & -s_{12}\cdot s_{13} \end{pmatrix}$

- new! · USE POSITIVITY: { + really easy to use; − need to prove signs } ⟶ this is all written in paper w/ Lenny

or

- new! · USE NEWTON POLYTOPES: { + works in char p; − hard to use } ⟵ this is in-progress (cf. Shenman-Zaslow)

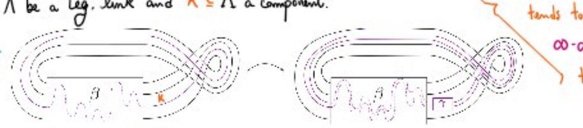
## In more generality: $\vartheta$-loops

← generalizes the loops constructed above to many more links

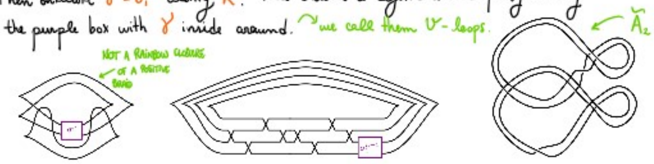
the conclusion is that the PURPLE BOX monodromy tends to have $\infty$-order on the DGA!

Let $\Lambda$ be a Leg. link and $K \subseteq \Lambda$ a component.

E.g.




Then satellite $\gamma = \sigma_1^n$ along K. This creates a Legendrian loop by moving the purple box with $\gamma$ inside around. ↷ we call them $\vartheta$-loops.




## Exceptions with $\vartheta$ of FINITE ORDER

in the cases in which finitely many fillings are known (and $\vartheta$ of finite order), aim is to find $\vartheta$-orbital structure!

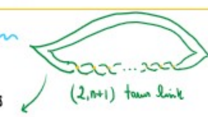
**Thm.** (C.-Hughes & Ng)

- $A_n$ case: there are $C_{n+1}$ fillings known and $\vartheta$ has

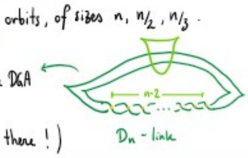
$$\boxed{\frac{1}{n}C_{n-2} + \frac{1}{2}C_{n/2-1} + \frac{2}{3}C_{n/3-1}}$$ orbits, of sizes $n$, $n/2$, $n/3$.

Catalan # number

Euler's identity for continuants ⟵




- $D_n$ case: the $\vartheta$-loop always has order n on the DGA

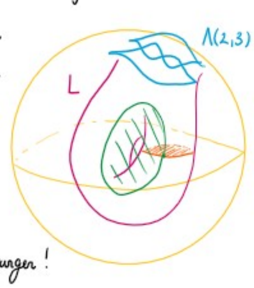



(The $E_6 = (3,4)$ and $E_8 = (3,5)$ follow from Kalman, $E_7$ still there!)

## §3. $\infty$'ly many fillings II: Lagrangian skeleta & Weaves

A geometrization of cluster algebras: a Lagr. skeleton for $(\mathbb{D}^4, \Lambda)$ ← Weinstein pair

STRATEGY: Suppose $L \subseteq (\mathbb{D}^4, \omega_{st})$ is a Lagrangian filling and ∃ Lagr. 2-disks $\mathbb{D}^2 \subseteq \mathbb{D}^4 \setminus L$ with boundary in L. Then we can SURGER THE $\mathbb{D}^2$'s.



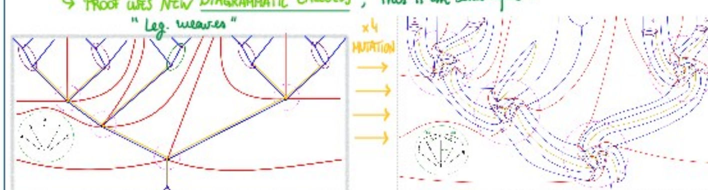

(a) **Thm.** (C'20): This skeleton exists if $\Lambda$ is a max-tb algebraic link.

(b) Still, iterating surgeries typically creates immersed curves $\Longrightarrow$ hard to find $\mathbb{D}^2$'s to surger!

**Thm**: (C.-Zaslow Spring '20) There exist Legendrian links $\Lambda \subseteq (\mathbb{S}^3, \xi_{st})$ for which the surgery strategy can be implemented. In particular, with a quiver $Q_\Lambda$ of infinite mutation type ($\Longrightarrow$ $\infty$'ly many Lagr. fillings)

↳ PROOF USES NEW DIAGRAMMATIC CALCULUS, this is an example:

"Leg. weaves"




## §4. ADE Conjecture

(see also BCFG cases!) ← arXiv 2009.06737

**Conj.** (2020) Let $\Lambda \subseteq (\mathbb{S}^3, \xi_{st})$ be a closure of a positive braid and $Q_\Lambda$ connected.

Then

A (i) $\Lambda \cong \Lambda(A_n)$ and $\Lambda$ has precisely $\frac{1}{n+2}\cdot\binom{2n+2}{n+1}$ Lagr. fill.

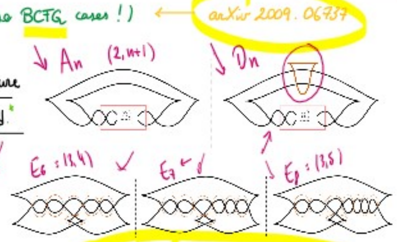
D (ii) $\Lambda \cong \Lambda(D_n)$ and $\Lambda$ has precisely $\frac{3n-2}{n}\cdot\binom{2n-2}{n-1}$ Lagr. fill.

E (iii) $\Lambda \cong \Lambda(E_6), \Lambda(E_7)$ or $\Lambda(E_8)$ with 833, 4160 and 25080 Lagr. fillings.

OR

(iv) $\Lambda$ has $\infty$'ly many Lagr. fillings.

Constructive bound at least 1 filling per cluster (J. Hughes (D type), B.H. An - Y. Bae - E. Lee (ADE type))




## SUMMARY OF TECHNIQUES

they all prove $\infty$'ly Lagr. fill.

**Direct methods** (C.-Gao '20):

- Yield the strongest results (e.g. $PSL_2\mathbb{Z}$)
- they are hard to prove ⇝ categorical faithfulness? (see C. Fraser & B. Keller)

**Lagr. skel & weaves** (C.-Zaslow Spring 20):

- only using the mutation class of $Q_\Lambda$, is it general?
- the diagrams can get seriously complicated
  - rel^n to Soergel calc. & Charact. varieties
  - J. Hughes for weaves in the $D_n$-case — can always mutate!

**Floer theory** (C.-Ng Fall'20):

- the only method that can tackle links which are not positive.
- required developing the theory / $\mathbb{Z}$ ⟶ or understand NEWTON POLYTOPES for a char 2 argument

**Cluster algebras** (Gao-Shen-Weng Fall'20):

- immediate if the DT transformation has $\infty$ order
- the case of non-positive links is a mystery. ⟶ Augmentation stacks are cluster? Hol. Sympl.?

## Future directions (of many!)

**1.** Let $\Lambda \subseteq (S^3, \xi_{st})$ be such that $Aug(\Lambda)$ cluster.

Consider $\mathcal{E} : \left\{ \begin{array}{c} \text{Lagr. fillings} \\ \text{of } \Lambda \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{clusters} \\ \text{of Aug} \end{array} \right\}$.

$L \longmapsto \mathcal{E}_L$

exc. surjective in finite type

Currently no examples where INJ. or SURJ. of $\mathcal{E}$ known.

**2.** Geography & Botany: study the map

$\left\{ \begin{array}{c} \Lambda \text{ Legendrian link} \\ \text{in } (M, \xi) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{cluster} \\ \text{algebras} \end{array} \right\}$

with $Aug(\Lambda)$ cluster

$\Lambda \longmapsto \mathbb{C}[Aug(\Lambda)]$

Double Bruhat cells ✓

Finite type ✓

Affine type ?

Quantum cluster alg. ?

**3.** Geometrization of algebraic features

(i) $\mathbb{C}[Aug(\Lambda)]$ cluster algebra, $f \in Mod(\mathbb{C}[Aug(\Lambda)])$

cluster modular group

$\boxed{\text{Is } f \text{ induced by Legendrian isotopy ?}}$

possibly more general ?

braid symmetry on $Gr(k,n)$ [CG]

$DT^2$ [GSW] (sometimes!)

Coxeter mutation [ABL]

$\boxed{\text{BASIS!?}}$

(ii) The $q$-parameter ?

(iii) Cluster $\mathcal{A}$- & $\mathcal{X}$-coord. in Aug & Sheaves ?

**4.** Develop & Standarize Connections to:

(i) Homological mirror symmetry

can we recover $\Lambda = \partial \mathbb{C}^2$ from $Aug(\Lambda)$ ?

(ii) Exact WKB analysis

WKB lines as Morse flow trees

(See also "spectral networks")

(iii) what is "holomorphic symplectic" topology ?

$K_2$-Lagr. in $Aug(\Lambda)$ ?

(iv) Singularities of plane curves

[FPST]

$\mathbb{R}$-Morsif.

natural cluster algebra

# THE END

Thank you !