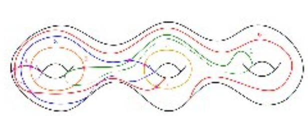
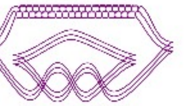
AMS Sectional Spec Session Talk

Saturday, April 17, 2021 7:41 AM

Legendrian Knots & Lagrangian fillings





R. Casals (UC Davis) @

Special Session of AMS Sectional Meeting '21

(1) 2001.01334 w/ H. Gao

(2) 2007.04943 w/ E. Zaslow

2009.06737 (3)

2101.02318 (4) w/ L. Ng

(5) 2012.06931 w/ E. Gorsky, M. Gorsky, J. Simental + in progress

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The problem : CLASSIFICATION of LAGRANGIAN FILLINGS — up to Hamiltonian isotopy, compact supp.

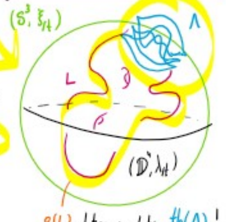
Today's setup : Fix Legendrian link $\Lambda \subseteq (S^3, \xi_{st})$ and study embedded exact Lagrangians $L \subseteq (\mathbb{D}^4, \lambda_{st})$ such that $\partial L = \Lambda \subseteq (\partial\mathbb{D}^4, \lambda_{st}|_{\partial\mathbb{D}^4})$ with cylindrical end near Λ.

The overarching strategy is :

1. Construct Lagrangian fillings : front surgeries (pinching seq.), generating functions, weaves, plabic graphs, divides & more.
2. Show that they are distinct : Legendrian DGA and augmentations, generating family homology, microlocal sheaves, cluster str. and more.
3. Prove that these are all : use moduli of J-holomorphic curves (w/ evaluation map) and argue in ad hoc manner.

this is starting to look good !

new ideas or arguments needed !



$g(L)$ determined by $tb(\Lambda)$!

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① **Constructing Lagrangian fillings** : consider $\Lambda \subseteq (S^3, \xi_{st})$ which are (-1)-closures of a braid $\beta \in Br_N$ s.t. $\beta \cdot \Delta^2 \in Br_N^+$. ← a case where we can currently say something

Pinching Sequences : (EHK, Pan)

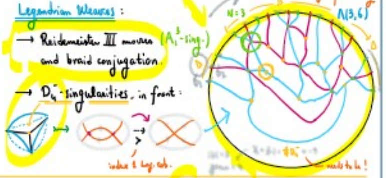
crucial !!! → Create Lag. cob. L by graphing Leg. isotopies.

→ Use index-1 Lag. cob. given by : in front, in Lag. proj.

Legendrian Weaves :

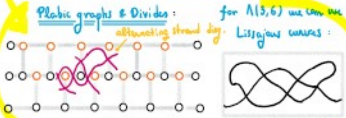
→ Reidemeister III moves (A_1^3-sing.) and braid conjugation.

→ D_4^--singularities, in front : index 1 Lag. cob.



Plabic graphs & Divides : alternating strand diag.

for $\Lambda(3,6)$ we can use Lissajous curves :



Here we use $T^\infty\mathbb{R}^2 \xrightarrow{Lag.} \mathbb{R}^2$, so all angles permitted but need co-orient.

From $\mathbb{R}$-morsification of $f(x,y) = x^3 + y^6$.

Three comments :

(i) Each method has its advantages and trickiness — complexity of combinatorics? interaction with DGA? interaction with sheaves? interaction with cluster str? — as far as I can tell, all have value !

(ii) The three are powerful enough to produce and detect "∞ Lagr. fillings".

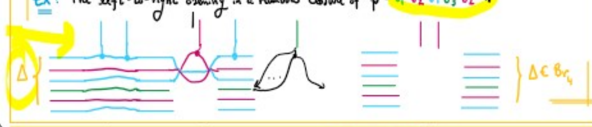
(iii) Translation between these 3 presentations is interesting !

Thm (Lagrangian Translations) In the context of $\Lambda(\beta) \subseteq (S^3, \xi_{st})$ as above :

(i) The Hamiltonian isotopy class of a Lagrangian filling obtained by a pinching sequence w/o R1 moves (and only R2 if conjugation) is represented by a Legendrian weave. (The converse also holds : from weave to pinchings.)

Furthermore, the translation is local and combinatorial.

Ex : The left-to-right opening in a rainbow closure of $\beta = \sigma_1^2\sigma_2\sigma_1\sigma_3\sigma_2^2$ is



(ii) The Hamiltonian isotopy class of a Lagrangian filling obtained as a conjugate surface of a plabic graph is represented by a Legendrian weave.

translations involve local models for D_4^- as a pinching sequence, movies of movies and satellites.

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② **Show that fillings are distinct** : gathering C.-Gao, C.-Zaslow, Gao-Shen-Weng, C.-Ng and in-progress

invariant	computability	detects "∞ Lagr. fillings"?	algebraic structure	What's next ?
Legendrian DGA and its augmentation spaces "FLOER THY"	· Absolute & Relative in pinching seq. ✓ · Leg. weaves ? · Plabic graphs ?	Yes, both directly and using the partial order $\Lambda_- < \Lambda_+$. * but only $\mathbb{Z}$ (not $PSL_2\mathbb{Z}$)	· Distinguish $\varepsilon_L : A_\Lambda \longrightarrow \mathbb{Z}[H_1(L)]$ by maximizing over $\mathbb{Z}$. · Distinguish by area of Newton polytope	· Name cluster A- and X-coordinates Floer theoretically. · Study cluster & hol. symp. str. beyond positive braids. · Develop a high dim thy of fillings
Moduli of constructible sheaves with $\mu supp$ in Λ "SHEAF THY"	· Absolute ✓ if braid in pinching seq. ? (Relative ?) · Leg. weaves ✓ · Plabic graphs ✓	Yes directly and for graphs of Leg. isotopies. (Not yet with partial order.) * known cases have most have $\beta \in Br^+$.	· Ad hoc arguments using the ring $\mathbb{C}[M]$ · Distinguish using quiver of $H_1(L)$ inter. · Distinguish using cluster structures (DT or by hand)	· Develop the relative thy with Lag. cob $\Lambda_- < \Lambda_+$. · Discover the right geometric structure that yields quantum cluster algebras. · Make sheaves "geometrically useful", e.g. J-hol. curves foliate → classif.

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③ **Prove that we have all** : the only Legendrian link $\Lambda \subseteq (S^3, \xi_{st})$ for which we have a non-empty complete classification is the standard unknot Λ_{st}. This is due Eliashberg-Polterovich ('96) via studying simple hypersurfaces & char. fol.

→ What's a good next candidate ? → but [EP] technique does not apply !

↳ ADE Conjecture ('20) hints towards certain torus links : (2,n), n ≥ 1, (3,4), (3,5) and a few more.

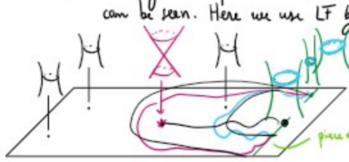
new! 2021 (2,2) Hopf link, (2,3) trefoil

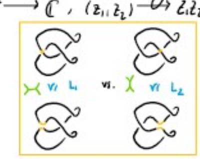
Thm. Let $L \subseteq (\mathbb{D}^4, \lambda_{st})$ be an embedded exact Lagrangian cylinder which coincides with $(\mathbb{D}^2(q_1,q_2) \times \{0\}) \cup (\{0\} \times \mathbb{D}^2(p_1,p_2))$ outside a compact set. Then L is Hamiltonian isotopic to one of the two Lagrange surgeries of the Lagrangian 2-planes $\mathbb{D}^2(q) \times \{0\}$ and $\{0\} \times \mathbb{D}^2(p)$ at their unique transverse intersection.

That is, the max-tb Hopf link has precisely 2 embedded exact Lagrangian fillings.

Sketch of argument : (Note that both STEP I & II need new ideas if we try the trefoil !)

→ STEP I : Find a geometric presentation of $(\mathbb{D}^4, \lambda_{st})$ where all candidate fillings can be seen. Here we use LF by conics $\mathbb{C}^2 \longrightarrow \mathbb{C}$, $(z_1, z_2) \mapsto z_1 z_2$.





→ STEP II : given an arbitrary filling $L \subseteq (\mathbb{D}^4, \lambda_{st})$ build a compatible foliation by J-holomorphic conics. This is done by

(a) Stretching the neck along L with specified asymptotics near $\partial L = \Lambda$ ⟶ yields a foliation by broken conics in a partial compactification

(b) Using a Levi-flat hypersurface near (a piece of) (S^3, ξ_{st}) to contain the curves. □

the compatibility comes from cylinder being foliated by geodesics

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THE END

Thank you !