**Introduction**

**Legendrian Knots in $\mathbb{R}^3$**

Def: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field $\xi^3 = \ker \left\{ \partial_x - y \partial_z \right\}$. It compactifies to $(\mathbb{S}^3, \xi^3)$.

Def: A knot $K \subset (\mathbb{R}^3, \xi^3)$ is **Legendrian** if $TK \subset \xi^3$.

**Lagrangian Fillings**

Def: A Lagrangian filling $L \subset (\mathbb{R}^4, \omega_{std})$ is an embedded exact Lagrangian surface in $\mathbb{R}^4$ with boundary $\partial L = \Lambda$ in $\mathbb{R}^3$.

**Salient Facts**

1. A $\Lambda$ might or might not have a Lagrangian filling.
2. If a filling $A$ then $g(L) = g(A)$.
3. (Chekanov-Rochon 1996) Let $\Lambda = \Lambda_{reg}$ be the smooth standard unknot. Then $\mathcal{L}^1$-filling (the flat disk) up to Hamiltonian isotopy.
4. Lagrangian fillings are the objects of $\mathcal{W}(\mathbb{C}^2, \Lambda)$, the wrapped Fukaya category shifted at $A$.

(Amoretti, Seidel)
1.2. Braid varieties I

A class of affine algebraic varieties which are useful to apply topology to $\mathcal{A}_n$.

Consider $B_+(F) = \left( \begin{array}{c} \vdots \\ \vdots \\ \ddots \end{array} \right) \in \text{GL}_n(\mathbb{C}[F])$.

Defn: Let $\beta \in B_n$ be a positive braid word $\beta = \beta_1 \ldots \beta_n$, the braid variety associated to $\beta$ is:

$X(\beta) = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : B_{\beta_1}(x_1) \ldots B_{\beta_n}(x_n) \text{ is upper-triangular} \} \subset \mathbb{C}^n$.

Example: $X(\beta^+) = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 1, x_1 \geq x_2 \geq x_3 \geq 0 \} \subset \mathbb{R}^3$.

Thm: $X(\beta)$ is independent of the choice of braid word $\beta \in B_n^+$.

In fact $X(\beta)$ is smooth and has $T$-action s.t.

$X(\beta)/T$ is smooth, holomorphic symplectic and its coordinate ring admits a cluster $A$-structure.

Example: $B_{(2,1)} B_{(1,2)} B_{(1,2)} B_{(1,2)} B_{(1,2)} B_{(1,2)} B_{(1,2,2)} B_{(1,2)} B_{(1,2)}$.

Thm: $f$ diagrammatic calculus to study the category of CORRESPONDENCES or BRAID VARIETIES.

The main protagonist is $6, 6 \mapsto 6'$ (nil Hecke move).
§ 3. Postroids in $\text{Gr}(k,n)$ -- or, "dude, where is my braid?"

There exists a stratification $\text{Gr}(k,n) = \bigcup_{u \in S_n, u \leq w} \Pi_{u,w}$ with a unique open stratum $\Pi_{w, w}^* \cong \text{proj of Richardson}$

$$\Pi_{w, w}^* = \{ V \in \text{Gr}(k,n) \mid \text{cone of cyclic projective monoid of } f \in \text{int cyclic comb. class} \}$$

**Example:** $(k,n) = (2, 5)$, $\Pi_{w, w}^* = \text{Gr}(2, 5) \setminus \{ \Delta_1, \Delta_2 \}$, $\Delta_1 \cdot \Delta_2 = 0$

with $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$, a trefoil knot.

The underlying braid is $\beta = wu = 6663, 66663 \rightarrow X(\beta) \not\cong \Pi_{w, w}^*$.

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§ 4. Braid varieties II -- negative crossings (RI) & Markov stabilizations

The two *take home messages* are:

1. Let $y \in \Theta_n$ be a positive word $\beta$. Then $X(y)$ affine variety $X(y)$ and a set of removal 0-cycles $V(y)$ s.t. $X(y)/V(y) \cong X(\beta)$.

2. Let $y \in \Theta_n$ be a positive word $\beta$. Then $X(y)/V(y) = X(\beta)$.

**Case study:**

$$\Pi_{w, w} = X(\Theta_n(\mu_{w}) \Delta_{w})/\mu \cong X(J_{\mu}(f)) \times \mathbb{C}^n$$

**Question:**

How is the pair $(X(y), V(y))$ built? Construct a DG-algebra $A(y)$ and find a good model for $H^2(A(y))$.

Let $y \in \Theta_n$ be a positive word $\beta$. The DG-algebra $A(y)$ is built as follows:

- freely graded commutative generated by $Y_j$ degree 1, $\zeta_i$ degree 0, $\omega_k$ degree 1
- $A(y)$ also $R_{2*}$-filtered
- $A(y)$ is an operad in $\mathbb{C}^*$
- differential given by $\Theta_j(\mu_{w})$ products counting $o^2 \cdot \mu_{w} = o \cdot \mu_{w} \cdot \mu_{w}$

**Example:** $J_{\mu}(f)$ a good, $X(J_{\mu}(f)) \cong \text{Gr}(3, 7)$.
**Theorem:** Let $\gamma \in \Theta_n$ be a pure twin word $\beta$. Then $\delta : A(\gamma) \rightarrow A(\gamma)$ satisfies $\delta^2 = 0$, and its cohomology $H^0(A(\gamma))$ is invariant under braid moves $\Delta$-conjugations. Moreover, $H^0(A(\gamma))$ is given by the affine variety $X(\gamma) = \{(z) : \delta_z - \delta_{x} = 0\}$ modulo $V(\gamma) = \langle \omega_n \rangle$.

**Example:** (Tscfo2) We discussed $X(\gamma')$ already, with braid $\gamma' = \cdots$. This gives the open preord $T_{1,5} \in Gr(4,5)$, and $A_n$-cluster $6^n$. What if we want $X(\delta, \delta', \delta'')$ instead?

Then $\delta : \gamma' \rightarrow (\omega_n)$ is given by

$$X(\gamma) = \{(z) : f(\{\gamma z, \delta z, \delta z, \delta z, \cdots \}, \delta z) = 0 \in \mathbb{C}^6$$

with $V(\gamma)$ generated by $t : (z_1, \cdots, z_t) \rightarrow (z_1 + t, z_2 - t, z_3, \cdots, z_t)$.

In fact, $B(\delta, \delta', \delta'')$ is a function on $z_1 + z_2$, so quasiant is direct.°

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**THE END**

Thank you!