MAT 22B: PROBLEM SET 1 SOLUTIONS

DUE TO FRIDAY OCT 4 AT 10:00AM

Abstract. These are the solutions to the first problem set for the Differential Equations Course in the Fall Quarter 2019. The Problem Set 1 was posted online on Friday Sep 27 and was due Friday Oct 4 at 10:00am via online submission.

Information: These are the solution for Problem Set 1 as designed by Professor R. Casals in his version of MAT22B Fall 2019. Problems are written in black and solutions in **blue**.

Purpose: The goal of this assignment is to practice the basic techniques for solving first-order differential equations. In particular, we would like to become familiar with method of **integrating factors** and **separation of variables**.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $t \in \mathbb{R}$ be a real number and $y(t)$ a differentiable real valued function. Find all the solutions to the following differential equations:

(a) $y'(t) = \frac{1}{1+t^2}$,

This equation can be integrated directly or solved by separation of variables:

$$\frac{dy}{dt} = \frac{1}{1+t^2},$$

then

$$dy = \int \frac{dt}{1+t^2} \implies \quad y = tan^{-1}(t) + C.$$

(b) $y'(t) = 3t^2 - 4 + \frac{1}{t} - \frac{7}{t^2}$, for $t > 0$,

Solve by separation of variables:

$$dy = (3t^2 + 4 + \frac{1}{t} - \frac{7}{t^2})dt.$$

Now take the anti-derivative of both sides such that

$$y(t) = t^3 + 4t + \ln t + \frac{7}{t} + C.$$

(c) $y'(t) = 5y(t)$,

Solve by separation of variables,

$$\frac{dy}{dt} = 5y.$$

The solution $y(t) \equiv 0$ is constant and treated separately. Now, for the rest, separate

$$\frac{dy}{y} = 5dt,$$

and taking the integral of both sides we get

$$\ln y = 5t + C, \quad C \in \mathbb{R}.$$

Solve for $y(t)$ we obtain

$$y = e^{5t} \cdot e^{C_1},$$

which we can write as $y(t) = Ce^{5t}$, $C \in \mathbb{R}^+$ with let $C = e^{C_1}$.

(d) $y''(t) = -y(t)$.

Let your $y(t) = C \sin t$, and note that $\cos(t)$ works as well, snice if you take the derivative twice of $\sin(t)$ you get $sin(t)$ which is what we wanted. The general solution is any linear combination

$$y(t) = C_1 \sin(t) + C_2 \cos(t), \quad C_1, C_2 \in \mathbb{R}.$$

Problem 2. Let $t \in \mathbb{R}$, and consider the function

$$y(t) = \arccos(t) + 4t^2.$$

(a) Find a first order differential equation which has $y(t)$ as a solution.

Take the derivative such that

$$y'(t) = -\frac{1}{\sqrt{1-x^2}} + 8t.$$

(b) Find a second order differential equation which has $y(t)$ as a solution.

Take the derivative of the equation in Part (a) s.t.

$$y''(t) = \frac{x}{(1-x^2)^{\frac{3}{2}}} + 8.$$

Problem 3. (20 pts) Consider the following differential equation:

$$y'(t) - \frac{3y(t)}{t+1} = (1+t)^4, \quad t > 0.$$

(a) Find all the solutions to the differential equation above.

We solve the differential equation via integrating factors. Let your integrating factor be

$$\frac{d\mu(t)}{dt} = \frac{3}{t+1}\mu(t), \quad \mu(t) = (t+1)^{-3},$$

which will multiply the original ODE in both sides. The left side will result in the derivative of a product, via the chain rule:

$$\frac{d}{dt}[(t+1)^{-3}y(t)] = (1+t).$$

Multiply the ODE with $\mu(t)$ throughout and then you take the intergral of both sides,

$$(t+1)^{-3}y(t) = t + \frac{1}{2}t^2 + C, \quad y(t) = (t+1)^3(t + \frac{1}{2}t^2 + C).$$

(b) Find all the solutions which satisfy $y(0) = 3$.

Solve for the equation of $y(t)$ found in Part (a) for the initial condition $y(0) = 3$ to find the value of C. Indeed, if $t = 0$ then $C = 3$ so that the unique solution is

$$y(t) = (t+1)^3(t + \frac{1}{2}t^2 + 3).$$

(c) Are there any solutions $y(t)$ such that $y(0) = 3$ and $y(1) = 100$?

Since this is a first order differential equations we have only one constant $C \in \mathbb{R}$, and using the initial condition in Part (b) we found $C = 3$. We verify that for that solution, when $t = 1$ the value of the solution is $y(1) = 36$, and thus $y(1) \neq 100$, and thus there are no solutions satisfying $y(0) = 3$ and $y(1) = 100$.

(d) Are there any solutions $y(t)$ such that $y(0) = 3$ and $y(1) = 36$?

Yes, the solution in Part (b).

Problem 4. (20 pts) Consider the following differential equation:

$$y'(t) + 2ty(t) = t, \quad t > 0.$$

(a) Find all the solutions to the differential equation above.
We solve by via integrating factors. The integrating factor is $\mu(t) = e^{t^2}$. Multiplying through we get

$$e^{t^2}y'(t) + 2e^{t^2}y(t) = te^{t^2}, \quad \frac{d}{dt}[e^{t^2}y(t)] = te^{t^2}.$$

By integrating both sides we get

$$e^{t^2}y(t) = \frac{1}{2}e^{t^2} + C.$$

The formula for all solutions is thus

$$y(t) = \frac{1}{2} + Ce^{-t^2}.$$

(b) Let $y_1(t)$ be the unique solution such that $y_1(2) = 0.5$, $y_2(t)$ the unique solution such that $y_2(0) = 1$, and $y_3(t)$ the unique solution such that $y_3(0) = 2$. Compute the long term behaviours

$$\lim_{t\to\infty} y_1(t), \quad \lim_{t\to\infty} y_2(t), \quad \lim_{t\to\infty} y_3(t)$$

for these three solutions.

Using the answer from Part (a), solve the initial value problem to find the constant C for each $y_1(t), y_2(t) and y_3(t)$. For $y_1(2) = 0.5$, we find

$$\frac{1}{2} = \frac{1}{2} + Ce^{-4}$$

so that the value of the constant is $C = 0$ and hence,

$$y_1(t) = \frac{1}{2},$$

and since $y_1(t)$ is constant to 0.5, then

$$\lim_{t\to\infty} y_1(t) = \frac{1}{2}.$$

For $y_2(0) = 1$, $1 = \frac{1}{2} + C$, so $C = 0.5$, hence $y_2(t) = \frac{1}{2} + \frac{1}{2}e^{-t^2}$. Thus

$$\lim_{t\to\infty} y_2(t) = \frac{1}{2}.$$

Finally, for $y_3(0) = 2$, we get C=$\frac{3}{2}$ so $y_3(t) = \frac{1}{2} + \frac{3}{2}e^{-t^2}$. Taking the limit $\lim_{t\to\infty} y_3(t)$ we get

$$\lim_{t\to\infty} y_3(t) = \frac{1}{2}.$$

(c) Plot the graph of the functions $y_1(t), y_2(t)$ and $y_3(t)$.

This is depicted in Figure 1, where $y_1(t)$ is represented by the red line, $y_2(t)$ by the blue line, and $y_3(t)$ by the green line

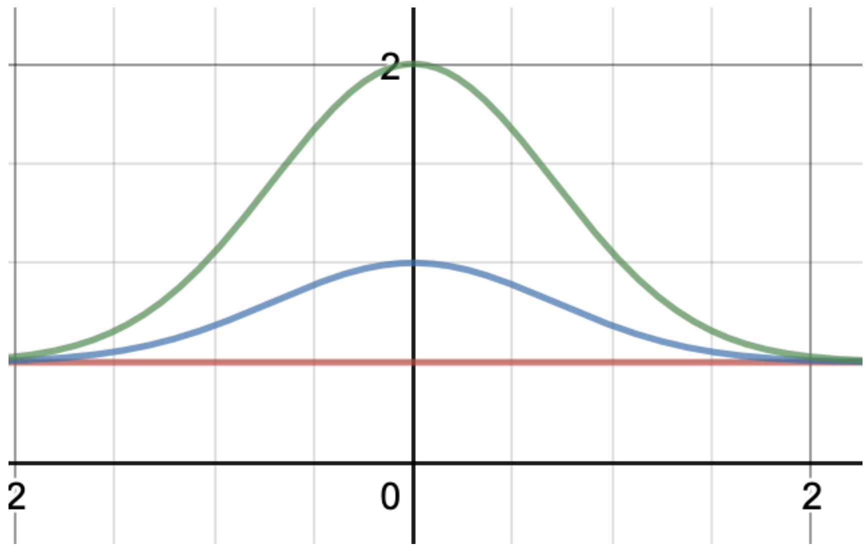

FIGURE 1. The solutions $y_1(t), y_2(t)$ and $y_3(t)$.

(d) Is there any solution $y(t)$ which tends to ∞ in its long term behaviour ?

No. This is due to the fact that the only part that changes is C which is just a constant, so if you consider the limit $\lim_{t\to\infty}$ the answer is always $\frac{1}{2}$, hence there is no solution $y(t)$ that tends to infinity.

Problem 5. (20 pts) Consider the following differential equation:

$$y'(t) = y^2 \cdot \frac{t-3}{t^3}, \quad t > 0.$$

(a) Find infinitely many the solutions to the differential equation above.

This equation can be solved by separation of variables:

$$\frac{dy}{dt} = y^2 \cdot \frac{t-3}{t^3}, \quad \frac{dy}{y^2 dt} = \frac{t-3}{t^3}.$$

Note that since we are dividing by $y(t)$, we take into account that the constant function $y(t) \equiv 0$ is *also* a solution. By integrating on both sides:

$$-\frac{1}{y} = -\frac{1}{t} + \frac{3}{2t^2} + C, \quad -\frac{1}{y} = \frac{2t - 3 - 2t^2 C}{2t^2}.$$

In conclusion, $y(t) = \frac{2t^2}{2t-3-2t^2C}$.

(b) Solve the initial value problem given by the above differential equation and the initial condition $y(1) = -1$.

Let us use $y(1) = -1$ to find the constant C. We obtain

$$-1 = \frac{2(1)^2}{2(1) - 3 - 2(1)^2 C},$$

and solving for C we get $C = \frac{1}{2}$. Thus, the solution would be $y(t) = \frac{2t^2}{2t-3-t^2}$.

(c) Is the constant function $y(t) \equiv 0$ a solution to the above differential equation ?

Yes, when substituted into the differential equation both sides vanish, and thus they are equal.

Problem 6. (20 pts) Consider the following differential equation:

$$y'(t) = 2t(1-y)^2, \quad t \in \mathbb{R}$$

(a) Find all solutions of the differential equation above.
Solve using separation of variables

$$\frac{dy}{dt} = 2t(1-y)^2,$$

but we take into account that the constant solution $y(t) \equiv 1$ must be considered separately, since otherwise we would be dividing by zero. Now, by separating:

$$\frac{dy}{(1-y)^2} = 2tdt$$

and taking the intergral of both sides:

$$\frac{1}{1-y} = t^2 + C,$$

which simplifies to

$$y(t) = \frac{t^2 + C - 1}{t^2 + C}.$$

The set of all solutions to the ODE is given by this formula above plus the constant solution $y(t) \equiv 1$.

(b) Does there exist a solution $y_1(t)$ such that $y_1(2) = 1$ and $y_1(3) = 1$?

Yes. The constant solution $y_1(t) = 1$ solves this initial value problem.

(c) Plot qualitatively the graph of six different solutions of the differential equation.

We use six different values for the constant C. We used the values 0 to 5 and depicted them in Figure 2, where zero is blue, 1 is red, 2 is green, 3 is purple, 4 is dashed red line, and 5 is the black line.

(d) Find the long-term behaviour of all solutions for the differential equation.

The long-term behaviour of all solutions is 1. Indeed, the limit is always $\lim_{t\to\infty} y(t) = 1$.

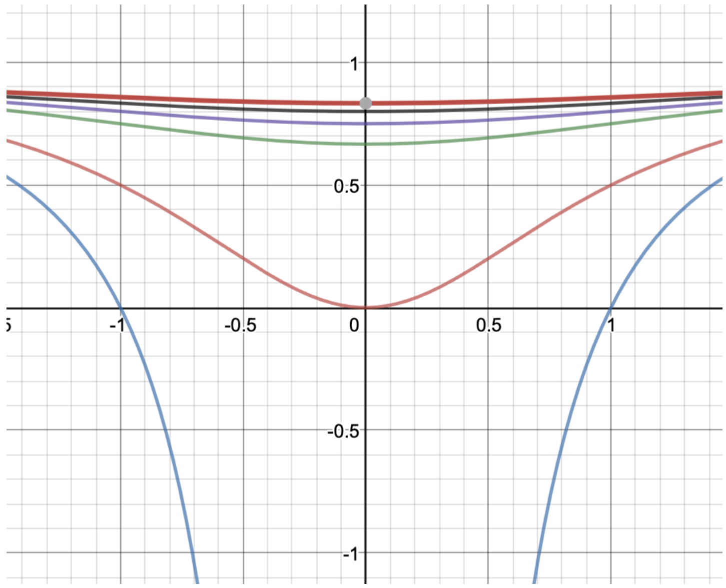

FIGURE 2. The solutions $y_1(t), y_2(t)$ and $y_3(t)$.

Problem 7. (20 pts) Consider the following differential equation:

$$y'(t) = y(t)(1 - y(t)), \quad t \in \mathbb{R}^+,$$

(a) Find all solutions of the differential equation above.

We solve this differential equation by using separation of variables:

$$\frac{dy}{dt} = y(t)(1 - y(t)) \implies \quad \frac{dy}{y(1-y)} = dt.$$

Note however that we are missing the constant solutions $y(t) \equiv 0$ and $y(t) \equiv 1$, which correspond to dividing by zero when rearranging. These solutions exist and we consider them separately. Now we proceed, from here the right side can be split using partial fractions:

$$\left(\frac{1}{y} - \frac{1}{1-y}\right) dy = dt.$$

Taking the integral on both sides we obtain

$$\ln\left(\frac{y}{1-y}\right) = t + C_1, \quad C_1 \in \mathbb{R},$$

and then we solve for $y(t)$, and thus $\frac{y}{1-y} = e^t \cdot e^{C_1}$. At this point, we can let $C = e^{C_1}$ and write

$$\frac{y}{1-y} = Ce^t, \quad C \in \mathbb{R}^+.$$

(b) Describe the possible long-term behaviour of a solution $y(t)$ in terms of its initial value $y(0)$.

If the initial value $y(0)$ is either 0 or 1 then its long-term behaviour is 0 and 1 respectively. If the initial value $y(0)$ satisfies $y(0) < 0$, then the long-term behaviour is $-\infty$. If $0 < y(0) \leq 1$, then the long-term behaviour is 1, and if $1 < y(0)$ then the solutions will also converge to 1, as $y(t) \equiv 1$ is a stable constant solution.

(c) Compare the differential equation above with the differential equation

$$y'(t) = y(t), \quad t \in \mathbb{R}^+,$$

as if they were two models for population growth, i.e. $y(t)$ is the size of a population at time $t \in \mathbb{R}$. Which of the two models is more accurate ?

The differential equation $y'(t) = y(t)$ has all solutions with long-term behaviour growing to ∞, which is not a realistic model for a population growth. Thus the model with $y'(t) = y(t)(1 - y(t))$, where some solutions converge to a finite population, is more realistic.

(d) Consider the differential equation

$$y'(t) = \lambda \cdot y(t)(1 - y(t)), \quad t \in \mathbb{R}^+, \lambda \in \mathbb{R}^+.$$

How do the solutions differ for distinct values of $\lambda \in \mathbb{R}$?

The overall behaviour is the same, and the only difference from the original differential equation is that the solutions converge at a different exponential rate, given that their exponential behaviour is now marked with $e^{t\lambda}$, in contrast to e^t.