MAT 22B: PROBLEM SET 2

DUE TO FRIDAY OCT 11 AT 10:00AM

Abstract. This is the second problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Oct 11 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice the basic techniques for qualitatively and numerically solving first-order differential equations. In particular, we would like to become familiar with method of **direction fields**, the study of **autonomous differential equations** and **Euler's Method**.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following differential equations, qualitatively plot their direction field in the (t, y)-plane:

(a) $y'(t) = y(t)(4 - y(t))$,

(b) $y'(t) = y(t)^2$,

(c) $y'(t) = \sin^2(t)$,

(d) $y'(t) = \dfrac{t^2}{1 - y(t)^2}$.

Problem 2. Consider the following differential equation:

$$y'(t) = y(t)^2 \cos(y(t)).$$

(a) Draw the direction field associated to this differential equation in the (t, y)-plane. (It should contain enough slopes such that the pattern is apparent.)

(b) Find the long-term behavior of *all* solutions $y(t)$ to this differential equation above in terms of their initial value $y(0)$.

(c) (Bonus) Give an example of a real-valued non-zero differentiable function $f(y)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, such that the differential equation $y'(t) = f(y(t))$ has at least a (continuous) interval worth of constant solutions, i.e. $\exists a, b \in \mathbb{R}$ such that the constant functions $y(t) \equiv c$ solve $y'(t) = f(y(t))$ for all $c \in (a, b)$.

Problem 3. (20 pts) Consider the following non-autonomous differential equation:

$$y'(t) = 2e^{t^2} + \frac{1}{5}\cos(y(t)), \quad t \geq 0.$$

(a) Plot qualitatively the direction field associated to this differential equation in the (t, y)-plane. The drawn direction field should contain drawn slopes such that the pattern is apparent, else use words to describe it.

(b) Show that the differential equation above does not admit any *constant* solution, i.e. a solution $y(t)$ is never a constant function.

(c) Describe the long-term behavior of *all* solutions to the differential equation.

Problem 4. (20 pts) Consider the following autonomous differential equation:

$$y'(t) = y(t)^4 - 6y(t)^3 + 11y(t)^2 - 6y(t).$$

(a) Plot qualitatively the direction field associated to this differential equation in the (t, y)-plane.

(b) Solve the initial value problem given by the above differential equation and the initial condition $y(0) = 3$.

(c) Find all the constant solutions to the differential equation.

(d) Describe whether each constant solution is stable, unstable or semistable.

Problem 5. (20 pts) Let $f(y)$ be the real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ depicted in Figure 1, and consider the autonomous differential equation $y'(t) = f(y(t))$.

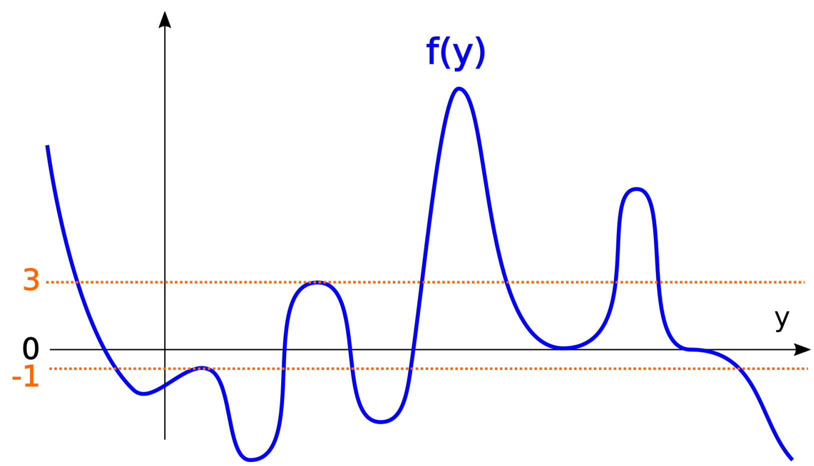

FIGURE 1. The function $f(y)$ for Problem 4.

(a) How many constant solutions does the above differential equation have ?

(b) Study whether the behaviour of each of the constant solutions of the differential equation $y'(t) = f(y(t))$ is stable, unstable or semistable.

(c) Discuss the long-term behaviour of *all* solutions $y(t)$ to this differential equation in terms of their initial condition $y(0)$.

(d) Consider also the autonomous differential equation:

$$y'(t) = f(y(t)) + 0.1.$$

Compare the behavior of its solutions with the behavior of the solutions of $y'(t) = f(y(t))$. In particular, find how many constant solutions does the differential equation $y'(t) = f(y(t)) + 0.1$ have.

Problem 6. (20 pts) The Steller's sea cow (*Hydrodamalis gigas*) was a sirenian discovered in 1741. Let us suppose that the population $y(t)$ of Steller's sea cow, in the thousands unit, in the year $1741 + t$ naturally followed the logistic growth:

$$y'(t) = 3y(t)(1 - y(t)/5).$$

(a) Show that the long-term behavior of *any* solution of the above differential equation is 5, i.e. the population of Steller's sea cow would converge to five thousand.

In fact, excessive hunting caused the extinction of the Steller's sea cow in 1768. Let us study this scientifically: we choose to model hunting in our differential equation by adding a term $-H$ to the differential equation, where $H \in \mathbb{R}^+$ is a positive constant modeling the rate of hunting:

$$y'(t) = 3y(t)(1 - y(t)/5) - H.$$

(b) Knowing that the species eventually extincted, deduce that $H > 3.75$.

(c) Suppose that $y(0) = 1000$, i.e. there were a million sea cows in 1741. Find the exact rate of hunting which led to their extinction in 1768, i.e. find the exact value of H such that $y(27) = 0$.

Problem 7. (20 pts) Consider the following initial value problem:

$$y'(t) = \sin(y(t) + t) - e^t, \quad y(0) = 4,$$

(a) Perform two steps of Euler's method with step $h = 0.1$ to obtain numerical approximations of $y(0.1)$ and $y(0.2)$.

(b) Give a numerical approximation to the value of the solution of the initial value problem at time $t = 0.3$, i.e. numerically approximate $y(0.3)$.

(c) Find an upper bound for the error in your approximation in Part (b).