MAT 22B: PROBLEM SET 2

DUE TO FRIDAY OCT 11 AT 10:00AM

ABSTRACT. This is the second problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Oct 11 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice the basic techniques for qualitatively and numerically solving first-order differential equations. In particular, we would like to become familiar with method of **direction fields**, the study of **autonomous differential equations** and **Euler's Method**.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following differential equations, qualitatively plot their direction field in the (t, y)-plane:

(a) $y'(t) = y(t)(4 - y(t))$,

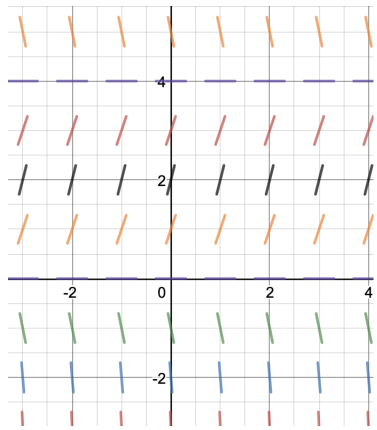


FIGURE 1. Direction field for $y'(t) = y(t)(4 - y(t))$

It should be clear that at $y = 0$ and $y = 4$ you would have no slope i.e. $y'(t) = 0$.

(b) $y'(t) = y(t)^2$,

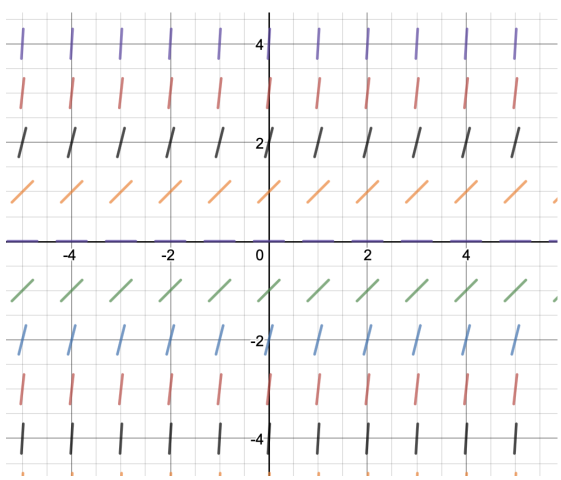


FIGURE 2. Direction Field for $y'(t) = y(t)^2$

A way to be able to draw this graph is noticing that all the slopes should be positive and the only area without slope should be 0.

(c) $y'(t) = \sin^2(t)$,

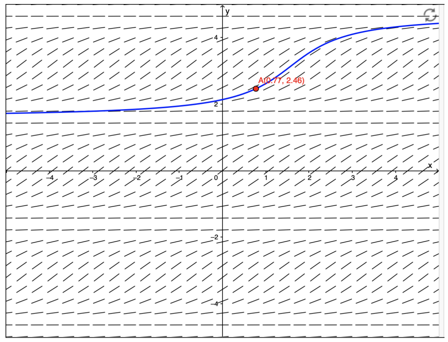


FIGURE 3. Direction Field for $y'(t) = \sin^2(t)$

(d) $y'(t) = \dfrac{t^2}{1 - y(t)^2}$.

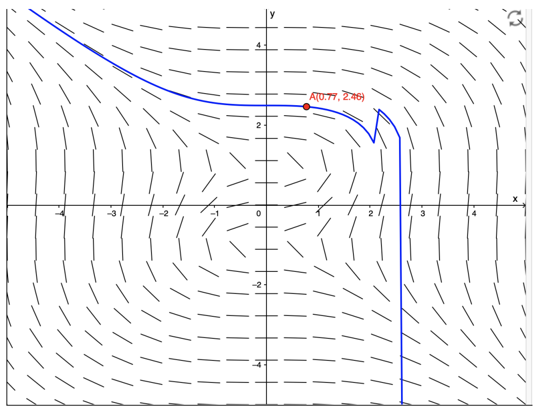


FIGURE 4. Direction Field $y'(t) = \frac{t^2}{1-y(t)^2}$

Problem 2. Consider the following differential equation:

$$y'(t) = y(t)^2 \cos(y(t)).$$

(a) Draw the direction field associated to this differential equation in the (t, y)-plane. (It should contain enough slopes such that the pattern is apparent.)

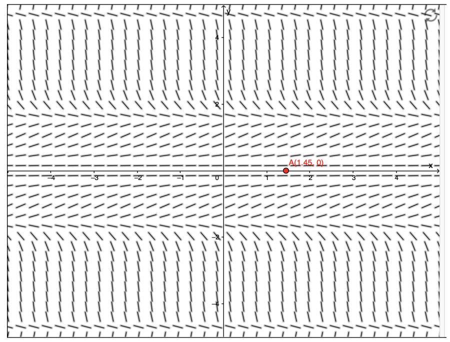


FIGURE 5. Direction Field for $y'(t) = y(t)^2 \cos(y(t))$.

(b) Find the long-term behavior of *all* solutions $y(t)$ to this differential equation above in terms of their initial value $y(0)$.

To find the overall behavior you will need to find the constant solutions, i.e. those solutions such that $y(t)^2 \cos(y(t)) = 0$. The constant solutions will be at $y(t) = 0$, which is a semistable solution, and $y(t) = \frac{n\pi}{2}$ where n$\in \mathbb{Z}$. Any solution between $\frac{n\pi}{2}$ and $\frac{(n+1)\pi}{2}$ will converge to $\frac{n\pi}{2}$ or $\frac{(n+1)\pi}{2}$, depending on the sign of $\cos(\frac{t\pi}{2})$ for $t \in [\frac{n\pi}{2}, \frac{(n+1)\pi}{2}]$, if it is positive then it will converge to $\frac{(n+1)\pi}{2}$, if negative then to $\frac{(n)\pi}{2}$.

(c) (Bonus) Give an example of a real-valued non-zero differentiable function $f(y)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, such that the differential equation $y'(t) = f(y(t))$ has at least a (continuous) interval worth of constant solutions, i.e. $\exists a, b \in \mathbb{R}$ such that the constant functions $y(t) \equiv c$ solve $y'(t) = f(y(t))$ for all $c \in (a, b)$.

Consider any smooth function $f(y)$ which is constant equal to 0 for $y \leq 0$ and non-zero for $y > 0$.

Problem 3. (20 pts) Consider the following non-autonomous differential equation:

$$y'(t) = 2e^{t^2} + \frac{1}{5}\cos(y(t)), \quad t \geq 0.$$

(a) Plot qualitatively the direction field associated to this differential equation in the (t, y)-plane. The drawn direction field should contain drawn slopes such that the pattern is apparent, else use words to describe it.

This is plotted in Figure 6. From the image you are able to see that the pattern is that the slope is always positive. This is because $\frac{1}{5}\cos(y(t))$ is bounded between $-1/5$ and $1/5$ and $2e^{t^2}$ is at least 2. Thus the sum is always positive.

(b) Show that the differential equation above does not admit any *constant* solution, i.e. a solution $y(t)$ is never a constant function.

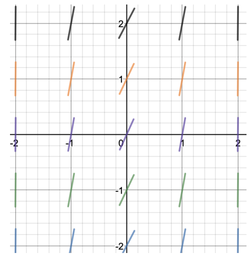


FIGURE 6. direction field plot

A simple way to solve this is knowing that a constant solution is given when $y'(t) = 0$. This implies

$$0 = 2e^{t^2} + \frac{1}{5}\cos y$$

, but the right hand side is always greater equal to $2 - 1/5$ by the argument above, thus never zero.

(c) Describe the long-term behavior of *all* solutions to the differential equation.

Any solution $y(t)$ goes to infinity as t tends to infinity because all the slopes are positive.

Problem 4. (20 pts) Consider the following autonomous differential equation:

$$y'(t) = y(t)^4 - 6y(t)^3 + 11y(t)^2 - 6y(t).$$

(a) Plot qualitatively the direction field associated to this differential equation in the (t, y)-plane.

The phase portrait for this equation is the following with constant solutions at $y(t) = 0, 1, 2, 3$:

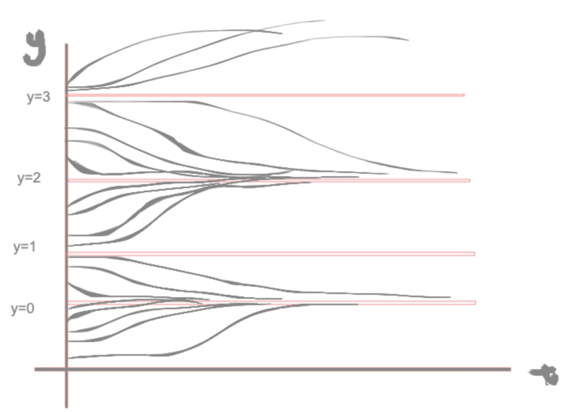


FIGURE 7. Phase diagram

(b) Solve the initial value problem given by the above differential equation and the initial condition $y(0) = 3$.

Plug in $y(0) = 3$ we get $y'(0) = 0$. So the solution is the constant solution $y(t) = 3$.

(c) Find all the constant solutions to the differential equation.

From your phase diagram, or by solving $y(t)^4 - 6y(t)^3 + 11y(t)^2 - 6y(t) = 0$, we obtain that the constant solutions are $y = 0, 1, 2, 3$.

(d) Describe whether each constant solution is stable, unstable or semistable.

This can be determined by the sign of $f(y) = y^4 - 6y^3 + 11y^2 - 6y$ near each zero, or by looking at the phase diagram, where the gray lines represents different initial conditions. At $y = 3$ we see that the solution diverge so $y = 3$ is a unstable. For $y = 2$ we se that the solutions converge from both sides so y=2 is a stable point. For $y = 1$ we see the exact same status as 3 so its unstable as well. Then for $y = 0$ it is clear that it is stable.

Problem 5. (20 pts) Let $f(y)$ be the real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ depicted in Figure 8, and consider the autonomous differential equation $y'(t) = f(y(t))$.

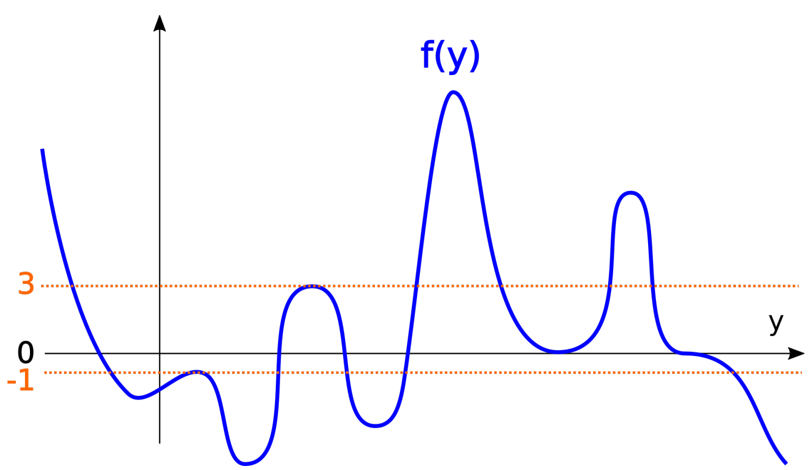


FIGURE 8. The function $f(y)$ for Problem 4.

(a) How many constant solutions does the above differential equation have ?

Constant solutions correspond to zeroes of $f(y)$, i.e. where the slope is zero. In this graph, it is when the graph passes through the y-axis, the horizontal axis. From left to right their is 6 places where it touches or crosses the y axis. Thus there are six constant solutions. We can name them $c_1, c_2, c_3....., c_6$.

(b) Study whether the behaviour of each of the constant solutions of the differential equation $y'(t) = f(y(t))$ is stable, unstable or semistable.

In the same picture you are able to check the stability of each of the fixed points i.e constant points. If it is $f(y) > 0$ then they converge to the right but if $f(y) < 0$ it goes to the left. In the following picture it is shown with black arrows. constant solutions c_1, c_3, c_6 are stable solutions as the arrows converge

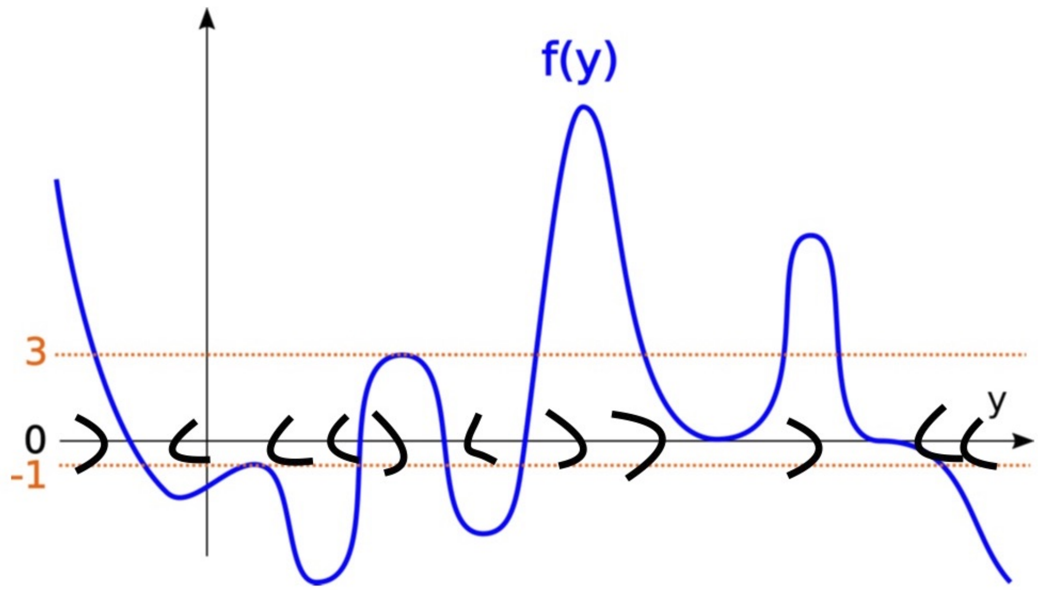


FIGURE 9. Graph showing the stability

to them. Solutions c_2, c_4 are unstable as the solutions diverge from them. The only semi stable solution is c_5 as points diverge from the right side, but converge from the left.

(c) Discuss the long-term behaviour of *all* solutions $y(t)$ to this differential equation in terms of their initial condition $y(0)$.

First lets say that the initial condition is $y(0) = k$. So if the constant solution $k < c_1$ then it will converge towards c1 so

$$\lim_{t\to\infty} y(0) = c_1.$$

this will be true as well for $c_1 < k < c_2$. When the solution for $y(0)$ falls $c_2 < k < c_3$ then we have

$$\lim_{t\to\infty} y(0) = c_3.$$

So we start to see patter depending where the initial conditions it will converge to the stable points. A special case will be when $c_4 < k < c_5$ from the right side

$$\lim_{t\to\infty} y(0) = c_5.$$

,but from the left side when $c_5 < k < c_6$ the limit would be

$$\lim_{t\to\infty} y(0) = c_6.$$

(d) Consider also the autonomous differential equation:

$$y'(t) = f(y(t)) + 0.1.$$

Compare the behavior of its solutions with the behavior of the solutions of $y'(t) = f(y(t))$. In particular, find how many constant solutions does the differential equation $y'(t) = f(y(t)) + 0.1$ have.

Compared to the original equation it will move up .1 so from the graph the only one that will be affected would be c_5 since it will come of the line so it will not be a constant solution no more. So

$$y'(t) = f(y(t)) + 0.1.$$

will only have 5 constant solutions and it will lose the semistable solution.

Problem 6. (20 pts) The Steller's sea cow (*Hydrodamalis gigas*) was a sirenian discovered in 1741. Let us suppose that the population $y(t)$ of Steller's sea cow, in the thousands unit, in the year $1741 + t$ naturally followed the logistic growth:

$$y'(t) = 3y(t)(1 - y(t)/5).$$

(a) Show that the long-term behavior of *any* solution of the above differential equation is 5, i.e. the population of Steller's sea cow would converge to five thousand.

There is various ways to solve this problem the easiest way is to plot the function $f(y) = 3y(1 - y/5)$, which is the right hand side of the differential equation:

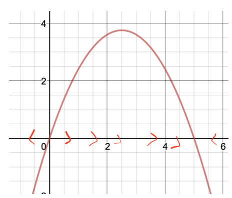


FIGURE 10. Stability Diagram

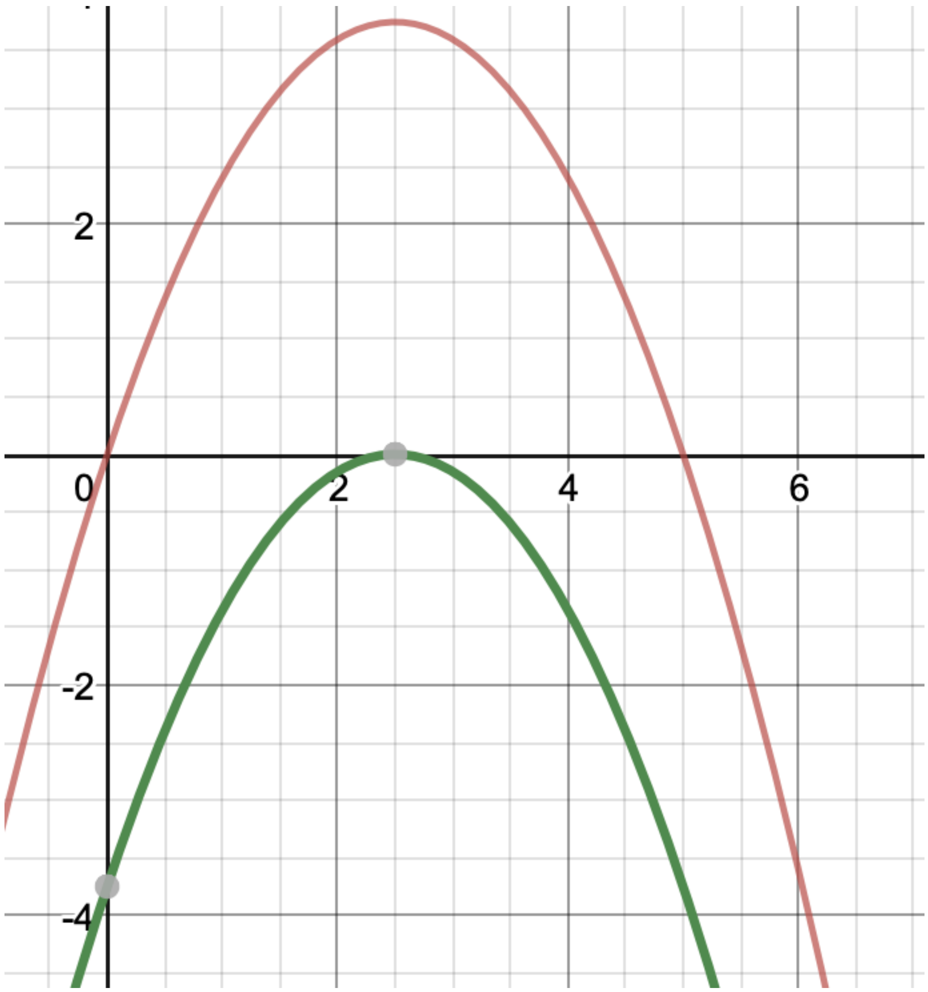
From this diagram we see there are two constant solutions for this equation and the stable one is when the population is 5 thousand. So any positive population would converge to that value.

In fact, excessive hunting caused the extinction of the Steller's sea cow in 1768. Let us study this scientifically: we choose to model hunting in our differential equation by adding a term $-H$ to the differential equation, where $H \in \mathbb{R}^+$ is a positive constant modeling the rate of hunting:

$$y'(t) = 3y(t)(1 - y(t)/5) - H.$$

(b) Knowing that the species eventually extincted, deduce that $H > 3.75$.

Suppose otherwise $H \geq 3.75$, then based on the graph, we know there exists $c > 0$ such that $y'(c) = 0$. Then $y = c$ is a constant solution, in contradiction to extinction. In short, for $H \geq 3.75$ the parabola $3y(1 - y/5) - H$ consists entirely of points with negative y-value, i.e. under the y-axis, this not constant solution and all leads to extinction.



(c) Suppose that $y(0) = 1000$, i.e. there were a million sea cows in 1741. Find the exact rate of hunting which led to their extinction in 1768, i.e. find the exact value of H such that $y(27) = 0$.

Using partial fractions:

$$\frac{1}{3y - 3y^2/5 - H} = \frac{A}{y-a} + \frac{B}{y-b},$$

where $A, B, a, b \in \mathbb{R}$ depend explicitly on H via the formula for roots of a second order polynomial. After separation of variables and integration on both sides we obtain:

$$|y-a|^A |y-b|^B = Ce^t$$

Plug in $y(0) = 1000$ and $y(27) = 0$:

$$|1000-a|^A |1000-b|^B = C$$

$$|a|^A |b|^B = Ce^{27}$$

The second equation allows to write C in terms of A, B, a, b, and thus C in terms of H. Then H is the unique value that solves the first equation, where

C has been substituted to depend on H as in the second equation.

Problem 7. (20 pts) Consider the following initial value problem:

$$y'(t) = \sin(y(t) + t) - e^t, \quad y(0) = 4,$$

(a) Perform two steps of Euler's method with step $h = 0.1$ to obtain numerical approximations of $y(0.1)$ and $y(0.2)$.

The Euler's method formula is given by

$$y_{n+1} = y_n + hf(t_n, y_n)$$

we are given that the step is h=0.1 and $y_0 = 4$ so for for the first step y(0.1) we get

$$y_1 = (4) + (0.1)(\sin(4)11)$$

$$y_1 = 3.824$$

then the next step is similar for y(0.2)

$$y_2 = 3.824 + (.1)(\sin(3.824 + 0.1) - e^{.1}$$

$$y_2 = 3.643$$

(b) Give a numerical approximation to the value of the solution of the initial value problem at time $t = 0.3$, i.e. numerically approximate $y(0.3)$.

You can continue using the euler method and solve for y_3 hence we get

$$y_3 = 3.643 + (0.1)(\sin(3.643 + 0.2) - e^{.2})$$

$$y_3 = 3.4566$$

(c) Find an upper bound for the error in your approximation in Part (b).

To find error aproximation you will need to the 2nd derivative so

$$y'(t) = \sin(y(t) + t) - e^t$$

the second derivative would be

$$y''(t) = \cos(y(t) + t)(y') - e^t$$

$$y''(t) = \cos(y(t) + t)(\sin(y(t) + t) - e^t) - e^t$$

since we want the upper bound we need to see the max value of this equation so

$$M = \frac{1}{2} - e^0(-1) = 1.5$$

we use the following values to solve for the error bound L=1, R=.1, t=.3, a=0. Now we can solver for the Error bound (E)

$$|E| = |\frac{(.1)(1.5)}{2}(e^{.3} - 1)|$$

so

$$|E| = .026239$$