MAT 22B: PROBLEM SET 3

DUE TO FRIDAY OCT 18 AT 10:00AM

Abstract. This is the third problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Oct 18 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice the basic techniques for solving second-order differential equations. In particular, we would like to become familiar with characteristic equations, damped oscillations and the method of undetermined coefficients.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following differential equations, find all solutions:

(a) $y''(t) = 0$,

(b) $y''(t) = t^3 - 3t^2 + 7$,

(c) $y''(t) = 144y(t)$,

(d) $y''(t) = -9y(t)$.

Problem 2. Consider the following differential equation:

$$y''(t) = 5y'(t).$$

(a) Find all solutions $y(t)$ to the differential equation above.

(b) Find two distinct solutions $y_1(t)$ and $y_2(t)$ such that $y_1(0) = 0$ and $y_2(0) = 0$.

(c) How many solutions are there such that $y(0) = 0$ and $y'(0) = 1$?

Problem 3. (20 pts) Consider the following second-order differential equation:

$$y''(t) + 3y'(t) = -2y(t).$$

(a) Suppose that $y(t) = e^{\lambda t}$ is a solution to the differential equation above. Find all possible values of $\lambda \in \mathbb{C}$.

(b) Find two linearly independent solutions of the differential equation above. In this context, two solutions $y_1(t), y_2(t)$ are linearly independent if $y_1(t)$ is not a constant multiple of $y_2(t)$.

(c) Use superposition to find *all* solutions to the differential equation.

(d) Find the unique solution $y_3(t)$ such that $y_3(0) = 0$ and $y_3'(0) = 5$.

(e) Describe the long-term behavior of *all* solutions to the differential equation.

Problem 4. (20 pts) Consider the following differential equation:

$$y''(t) - 6y'(t) + 10y(t) = 0.$$

(a) Find the characteristic equation associated to the differential equation above.

(b) Find *all* solutions to the differential equation above.

(c) Is there any constant solution $y_1(t)$?

(d) Find the long-term behaviour

$$\lim_{t \to \infty} y(t)$$

for all solutions $y(t)$ to the differential equation.

(e) Find the unique solution $y_2(t)$ satisfying $y_2(0) = 2$ and $y_2'(0) = 9$ and plot its graph. How many zeroes does it have ?

Problem 5. (20 pts) Let $\gamma \in \mathbb{R}$ be a positive real number and consider the damped system modeled by the following second-order differential equation:

$$y''(t) + \gamma y'(t) + 25y(t) = 0,$$

(a) Show that the long-term behaviour of *all* solutions is independent of γ.

(b) For which values of $\gamma \in \mathbb{R}^+$ does the above differential equation have oscillating solutions ? (i.e. solutions with infinitely many zeroes.)

(c) Classify the above damped system into underdamped, critically damped and overdamped in terms of γ, and qualitatively draw solutions corresponding to each of these three cases.

(c) Find all solutions to the above differential equation for $\gamma = 10$.

(d) For $\gamma = 10$, let $y_1(t)$ be a solution such that $y_1(0) = 0$ and $y_1(1) = 1$, and $y_2(t)$ a solution such that $y_2(0) = 1$ and $y_2'(0) = 0$. Compute the ration $y_1(t)/y_2(t)$ for $t \to \infty$ and deduce which solution converges faster to zero.

Problem 6. (20 pts) Let us consider the following damped system with external forced vibrations:

$$y''(t) + 2y'(t) - 35y(t) = \cos(4t).$$

(a) Find *one* solution to the above differential equation.

(b) Find *all* solutions to the differential equation by using the principle of superposition. Plot them qualitatively.

(c) Does there exist a solution $y_1(t)$ whose long-term behavior is infinite ?

(d) Let us increase our oscillating forced vibration from $\cos(4t)$ to $\cos(4t) + 9e^{6t}$. This yields the system

$$y''(t) + 2y'(t) - 35y(t) = \cos(4t) + 9e^{6t}.$$

Find all solutions to this differential equation.

Problem 7. (20 pts) Consider the following second-order differential equation:

$$y''(t) - 6y'(t) + 10y(t) = te^{5t},$$

(a) Find a particular solution $y_p(t)$ to the differential equation above.

(b) Find the general solution to the differential equation above.

(c) How many solutions $y(t)$ are there with $y(0) = 0$ and $y'(0) = 1$? If any, find them and qualitatively plot their graphs.

(d) Let us decide to approximate $e^{5t} \approx 1 + 5t + \frac{25}{2}t^2 + O(t^3)$ via a Taylor expansion and consider the system:

$$x''(t) - 6x'(t) + 10x(t) = t(1 + 5t + \frac{25}{2}t^2).$$

Find a particular solution $x_p(t)$ to this differential equation.

(e) Compare the solution $y_p(t)$ obtained in Part (a) with the particular solution $x_p(t)$. Is one related to the Taylor expansion of the other ? Do their long-term behaviors coincide ?